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POINT-COUNTABLE k-NETWORKS, cs^*-NETWORKS AND α_4-SPACES[1]

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ABSTRACT. In this paper the authors give the relations between point-countable k-networks and cs^*-networks by virture of α_4-spaces, and prove some mapping theorems on spaces with point-countable p-k-networks.

Since Burke, Gruenhage, Michael and Tanaka[3,4,7,20] established the fundamental theory on point-countable covers in generalized metric spaces, many topologists have discussed the point-countable covers with various characters. Meanwhile, the conceptions of k-networks, cs*-networks and p-k-networks were introduced. The study for relations among certain point-countable covers has become one of the most important subjects in general topology. In this paper, we shall consider the connections between point-countable k-networks and cs^*-networks, prove that α_4-spaces with point-countable k-networks have point-countable cs^*-networks, and obtain some closed mapping theorems about spaces with point-countable p-k-networks, which generalized some results in [7,13,14].

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In this paper, all spaces are assumed to be regular and T_1, and all mappings are continuous and surjective. We recall some basic definitions.

Definition 1. Let X be a space, and $\mathcal{P}$ a cover of X. Put

$$\mathcal{P}^{<\omega} = \{\mathcal{P}' \subset \mathcal{P} : |\mathcal{P}'| < \omega\}.$$

(1) $\mathcal{P}$ is a k-network [15] if, whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathcal{F} \subset U$ for some $\mathcal{F} \in \mathcal{P}^{<\omega}$.
(2) $\mathcal{P}$ is a p-k-network [7] if, whenever $K \subset X\backslash\{y\}$ with K compact in X, then $K \subset \cup\mathcal{F} \subset X\backslash\{y\}$ for some $\mathcal{F} \in \mathcal{P}^{<\omega}$
(3) $\mathcal{P}$ is a cs^*-network [6] if $\{x_n\}$ is a sequence converging to $x \in X$ and U is a neighborhood of x, there exists a $P \in \mathcal{P}$ such that $\{x\} \cup \{x_{n_i} : i \in N\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$.
(4) $\mathcal{P}$ is a wcs^*-network [11] if $\{x_n\}$ is a sequence converging to $x \in X$ and U is a neighborhood of x, there exists a $P \in \mathcal{P}$ such that $\{x_{n_i} : i \in N\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$
(5) $\mathcal{P}$ is a p-wcs^*-network if $\{x_n\}$ is a sequence converging to $x \in X$ and $y \neq x$, there exists a $P \in \mathcal{P}$ such that $\{x_{n_i} : i \in N\} \subset P \subset X\backslash\{y\}$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$:

p-k-networks were studied in [7], where they were labeled $(1.4)_p$. cs^*-networkis, wcs^*-networks, p-k-networks and p-wcs^*-networks were studied in [20], where they were labeled (C_1), (C_2), (C_3) and (C_3)' respectively.

Definition 2. [9] Call a subspace of a space a fan (at a point x) if it consists of a point x, and a countably infinite family of disjoint sequences converging to x. Call a subset of a fan a diagonal if it is a convergent sequence meeting infinitely many of the sequences converging to x and converges to some point in the fan.

(1) A space X is an α_4-space [2] if every fan at x of X has a diagonal converging to x.
(2) S_ω is a fan without a diagonal.

Definition 3. [5] For a space X and $x \in P \subset X$, P is a sequential neighborhood at x in X if, whenever $\{x_n\}$ is a sequence converging to x in X, then $x_n \in P$ for all but finitely many $n \in N$.

Definition 4. Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of X which satisfies that for each $x \in X$,

(1) $\mathcal{P}_x$ is a network of x in X,
(2) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is an sn-network[19] for X if each element of $\mathcal{P}_x$ is a sequentail neighborhood of x in X. $\mathcal{P}$ is a weak base [1] for X if, whenever G is subset of X such that for each $x \in G$, $P \subset G$ for some $P \in \mathcal{P}_x$, then G is open in X. A space X is an snf-countable space if X has an sn-ntework $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable. A space X is a gf-countable space if X has a weak base $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable.

We introduce the following symbols. For a subset collection $\mathcal{F}$ of a space X, let $\text{Int}_s(\mathcal{F}) = \{x : \cup\mathcal{F}$ is a sequential neighborhood at x in $X\}$. A subset collection $\mathcal{F}$ of X is an sn-cover of A if $A \subset \text{Int}_s(\cup\mathcal{F})$. For a cover $\mathcal{P}$ of a space X, put

(A): If $x \in U \in \tau(X)$, there exists $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $x \in \text{Int}_s(\cup\mathcal{F}) \subset \cup\mathcal{F} \subset U$.
(B): If $x \in U \in \tau(X)$, there exists $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $x \in \text{Int}_s(\cup\mathcal{F}) \subset \cup\mathcal{F} \subset U$, and $x \in \cap\mathcal{F}$.

It is clear that [9] gf-countable spaces $\Rightarrow$ snf-countable spaces $\Rightarrow$ α_4-spaces $\Rightarrow$ spaces without any closed copy of S_ω, and that

$$\begin{array}{ccccccc}
\text{weak base} \Rightarrow sn\text{-network} \Rightarrow & (B) & \Rightarrow & cs^*\text{-network} & & \\
& \Downarrow & & \Downarrow & & \\
& (A) & \Rightarrow & wcs^*\text{-network} & \Leftarrow & k\text{-network} \\
& & & \Downarrow & & \Downarrow \\
& & & p\text{-}wcs^*\text{-network} & \Leftarrow & p\text{-}k\text{-network}.
\end{array}$$

It is well known that for a space X,

(1) X has a point-countable weak base $\Leftrightarrow$ X is a gf-countable space with a point-countable sn-network $\Rightarrow$ X has a point-countable k-network[11].
(2) X has a point-countable sn-network $\not\Leftarrow$ X has a point-countable p-k-network, for example, the Stone-Čech compactification βN.
(3) X has a point-countable cs^*-network $\not\Rightarrow$ has a point-countable cover satisfying (A), for example, S_ω.
(4) X has a point-countable cs^*-network $\not\Leftarrow$ X has a point-countable cs^*-network, or X has a point-countable cover satisfying (A), for example, S_{ω_1} [20].
(5) X has a point-countable p-k-network $\not\Rightarrow$ X has a point-countable wcs^*-network, for example, let ψ^* be the well known Mrowka's space. Then ψ^* is a Moore space without any point-countable base. A Moore space has a point-countable p-k-network by Corollary 3.8 in [7].

In this paper we shall further study the relations among point-countable covers satisfying the conditions above.

Lemma 5. *Let X be an α_4-space, and $\mathcal{P}$ a point-countable wcs^*-network, then $\mathcal{P}$ has (A).*

Proof: Suppose the conclusion is false for some $x \in U \in \tau(X)$. For each countable $C \subset U$, let $\mathcal{P}(C) = \{P \in \mathcal{P} : P \cap C \neq \emptyset,\ P \subset U\} = \{P_i(C) : i \in N\}$. Put $C_0 = \{x\}$, then $P_1(C_0)$ is not a sequential neighborhood at x. There exists a sequence $\{x_{1n}\}$ in $U \backslash P_1(C_0)$ converging to x. Let $C_1 = \{x_{1n} : n \in N\}$, as $\mathcal{P}$ is a wcs^*-network of X, there is an $n_1 \in N$ and an infinite subet C_1' of C_1 such that $C_1' \subset \cup\{P_i(C_j) : 1 \leq i \leq n_1, 0 \leq j \leq 1\}$. Without losing generality, we can suppose $C_1' = C_1$, then $\cup\{P_i(C_j) : 1 \leq i \leq n_1, 0 \leq j \leq 1\}$ is not a sequential neighborhood at x. Repeating the process above, we can inductively choose $C_m = \{x_{nm} : n \in N\} \subset U$ such that $\{x_{nm}\}$ converges to x for each $m \in N$, and an increasing sequence $\{n_m\}$ with $C_m \subset \cup\{P_i(C_j) : 1 \leq i \leq n_m, 1 \leq j \leq m\} \backslash \cup \{P_i(C_j) : 1 \leq i \leq n_{m-1}, 0 \leq j \leq m-1\}$. The last condition implies that each $P \in \mathcal{P}$ meets only finitely many C_m.

Let $S = \{x_{nm} : m \in N, n \in N\} \cup \{x\}$, then S is a fan at the point x. Because X is an α_4-space, S has a diagonal converging to x, thus we can find $P \in \mathcal{P}$ and $P \subset U$ such that P meets infinitely many C_m, a contradiction.

The following Lemma is similar to Miščenko's Lemma [16].

Lemma 6. *If $\mathcal{P}$ is a point-countable cover of a space X, then every $A \subset X$ has only countably many minimal finite sn-covers by elements of $\mathcal{P}$.*

Proof: For every $\mathcal{F} \in \mathcal{P}^{<\omega}$, let $\mathcal{H}(\mathcal{F}) = \{H \subset X : \mathcal{F}$ is a minimal sn-cover of $H\}$. If there are uncountable many $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $A \in \mathcal{H}(\mathcal{F})$, then we can choose an $m \in N$ and an uncountable subset Λ of $\mathcal{P}^{<\omega}$ such that $|\mathcal{F}| = m$ and $A \in \mathcal{H}(\mathcal{F})$ for every $\mathcal{F} \in \Lambda$.

Suppose $\mathcal{R}$ is a maximal subset of $\mathcal{P}$ satisfying $\mathcal{R} \subset \mathcal{F}$ for uncountably many $\mathcal{F} \in \Lambda$, then $0 \leq |\mathcal{R}| < m$. Let $\Gamma = \{\mathcal{F} \in \Lambda : \mathcal{R} \subset \mathcal{F}\}$. If $\mathcal{F} \in \Gamma$, $A \not\subset \mathrm{Int}_s(\cup\mathcal{R})$. Choosing $x \in A \backslash \mathrm{Int}_s(\cup\mathcal{R})$, them there exists a sequence $\{x_n\}$ converging to x such that all $x_n \notin \cup\mathcal{R}$. Let $L = \{x_n : n \in N\}$, then L meets an element of $\mathcal{F}$. Because $\mathcal{P}$ is point-countable and Γ is uncountable, we can obtain $P \in \mathcal{P}$ such that $L \cap P \neq \phi$ and uncountably many elements of Γ contain P, so $P \notin \mathcal{R}$ and uncountably many elements of Γ contain $\mathcal{R} \cup \{P\}$. The last condition implies a contradiction.

Theorem 7. *The following are equivalent for a space X:*

(1) *X has a point-countable cover satisfying (A).*
(2) *X has a point-countable cover satisfying (B).*
(3) *X is an α_4-space (or an snf-countable space) with a point-countable cs^*-network.*
(4) *X is an α_4-space (or an snf-countable space) with a point-countable wcs^*-network.*

Proof: (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is obvious, and (4) $\Rightarrow$ (1) holds by Lemma 5, We only need to show that (1) $\Rightarrow$ (2). Let $\mathcal{P}$ be a point-countable cover satisfying (A). For each $\mathcal{F} \in \mathcal{P}^{<\omega}$, put $M(\mathcal{F}) = \{x \in X : \cup\mathcal{F}$ is a minimal sn-cover of $\{x\}\}$. It follows from Lemma 6 that if $x \in X$, then $x \in M(\mathcal{F})$ for only countable many $\mathcal{F} \in \mathcal{P}^{<\omega}$. For each $P \in \mathcal{P}$, let

$$P' = P \cup (\cup\{M(\mathcal{F}) : \mathcal{F} \in \mathcal{P}^{<\omega}, P \in \mathcal{F}\}).$$

Then $P' \subset \overline{P}$. In fact, for every $x \in M(\mathcal{F})$ and $P \in \mathcal{F}$, then $x \in \mathrm{Int}_s(\cup\mathcal{F})$, while $x \notin \mathrm{Int}_s(\cup(\mathcal{F}\setminus\{P\}))$, so there exists a sequence in P converging to x. Thus $x \in \overline{P}$, then $P' \subset \overline{P}$.

Let $\mathcal{P}' = \{P' : P \in \mathcal{P}\}$. For every $x \in X$, put $\mathcal{A}_x = \cup\{\mathcal{F} \in \mathcal{P}^{<\omega} : x \in M(\mathcal{F})\}$, then $\mathcal{A}_x$ is countable. Since $x \in P' \Leftrightarrow x \in P$ or $x \in M(\mathcal{F})$ with $P \in \mathcal{F}$, and $P \in \mathcal{A}_x$, $\mathcal{P}'$ is point-countable

For every $x \in W \in \tau(X)$, there exists $U \in \tau(X)$ such that $x \in U \subset \overline{U} \subset W$. Choosing $\mathcal{F}_0 \in \mathcal{P}^{<\omega}$ such that $x \in \mathrm{Int}_s(\cup\mathcal{F}_0) \subset \mathcal{F}_0 \subset U$. We can suppose $x \in M(\mathcal{F}_0)$. Let $\mathcal{F}_0' = \{P' : P \in \mathcal{F}_0\}$, then $x \in P'$ for every $P' \in \mathcal{F}_0'$. On the other hand, $x \in \cup\mathcal{F}_0' \subset \overline{\cup\mathcal{F}_0'} \subset \overline{U} \subset W$. $\mathcal{P}'$ satisfies (B).

Now, we give two interesting corollaries about Theorem 7. In view of Proposition 1.2 and Remark 1.3 in [20], we have

Lemma 8. *Let $\mathcal{P}$ be a point-countable cover of X, then $\mathcal{P}$ is a k-network (or a p-k-network) if and only if it is a wcs*-network(or a p-wcs*-network), and every compact subset of X is sequentially compact.*

Corollary 9. *For an α_4-space X, the following are equivalent:*

(1) *X has a point-countable k-network.*
(2) *X has a point-countable cs*-network, and each compact subset of X is sequentially compact.*

Corollary 10. *The following are equivalent for a k-space X, with a point-countable k-network:*

(1) *X is an α_4-space*
(2) *X contains no closed copy of S_ω.*
(3) *X is an snf-countable space.*
(4) *X is a gf-countable space.*

Proof: (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (3) by Theorem 3.13 in [9] and Theorem 7, and (1) $\Leftrightarrow$ (2) by Corollary 3.9 in [9].

Remark 11. Burke and Michael [3,4] proved that for a space X the following are equivalent:

(1) X has a point-countable base.
(2) X has a point-countable cover $\mathcal{P}$ such that, if $x \in U \in \tau(X)$, there exists $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $x \in (\cup\mathcal{F})^0 \subset \cup\mathcal{F} \subset U$, and $x \in \cap\mathcal{F}$.
(3) X is a k-space with a point-countable cover $\mathcal{P}$ such that, if $x \in U \in \tau(X)$, there exists $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $x \in (\cup\mathcal{F})^0 \subset \cup\mathcal{F} \subset U$.

But a space with a point-countable cover satisfying (B) can not be a space with a point-countable sn-network. By Remark 14(2) in [11], a gf-countable space X with a point-countable k-network (hence a space with a point-countable cover satisfying (B)) $\not\Rightarrow$ X has a point-countable sn-network. By Half-Disk Topology in [19, Example 78], there is a first countable and T_2-space X with a point-countable cover satisfying (A) such that X has not any point-countable cs^*-network, thus it shows that the regularity of the space in Theorem 7 is essential. Corollary 9 partly answers the following question posed by Chuan Liu[12]: Is a Fréchet space with a point-countable k-network which contains no closed copy of S_{ω_1} a space with a point-countable cs^*-network? Corollary 10 improves Theorem 1 in [13] and Theorem 20 in [14], and affirmatively answers the following question posed by Liu and Tanaka in [14]: Is a k-space with a σ-compact-finite k-network a gf-countable space if it contains no closed copy of S_ω?

Finally, we shall prove some closed mapping theorems on spaces with point-countable p-k-networks. It is well known that spaces with point-countable p-k-networks are preserved by perfect maps[7].

Theorem 12. *Let $f : X \to Y$ be a closed map such that X has a point-countable p-k-network, if one of the following holds, then Y has a point-countable p-k-network.*

(1) *X is a k-space.*
(2) *Each point of X is a G_δ-set.*
(3) *X is a normal, isocompact space.*
(4) *Each $Bf^{-1}(y)$ is Lindelöf.*

Proof: We only prove the result for (1), the other proof is similar to [11]. We note that every compact subset of X is metrizable in view of Lemma 8. If (1) holds, then X is a sequential space. By Lemma 2 in [11], each compact subset of Y is sequentially compact. So it suffices to show that Y has a point-countable p-wcs^*-network by Lemma 8. Let $\mathcal{P}$ be a point-countable p-k-network of X. For each $y \in Y$, choose $x_y \in f^{-1}(y)$ and put $A = \{x_y : y \in Y\}, \mathcal{P}^* = \{f(A \cap P) : P \in \mathcal{P}\}$, then $\mathcal{P}^*$ is a point-countable cover of Y. Let $S = \{y_n : n \in N\}$ be a sequence converging to $y \in Y, z \neq y$ and all $y_n \neq z$. Choosing $x_n \in f^{-1}(y_n) \cap A$, $x' \in f^{-1}(z) \cap A$, and $x \in f^{-1}(y) \cap A$. Then there exists a convergent subsequence T of $\{x_n\}$ in $X \backslash \{x'\}$[11]. $\mathcal{P}$ is a p-k-network for X, hence we can find out a $P \in \mathcal{P}$ and a subsequence Z of T such that $Z \subset P \subset X \backslash \{x'\}$, and $f(Z) \subset f(P \cap A) \subset Y \backslash \{z\}$, so this shows that $\mathcal{P}^*$ is a point-countable p-wcs^*-network for Y.

Corollary 13. *Let $f : X \to Y$ be a closed map with Lindelöf fibres such that X has a point-countable p-k-network, then Y has a point-countable p-k-network.*

Remark 14. M. Sakai[17] showed that there is a closed map from a space X in which every compact subset is finite onto the one-point compactification Y of ω_1. That X has a point-countable k-network, and Y has not a point-countable p-k-network. Let $f : X \to Y$ be a map. f is a compact-covering map if each compact subset of Y is the image of some compact subset of X. Let $f : X \to Y$ be a closed map. If X is a normal, isocompact space or each $Bf^{-1}(y)$ is Lindelöf, then f is a compact-covering map [11]. If X has a point-countable

k-network, and X is a k-space or each point of X is a G_δ-set, then f is also a compact-covering map[10,18]. But, if X has a point-countable p-k-network, and X is a k-space or each point of X is a G_δ-set, then f can not be a compact-covering map. In fact, let ψ^* be the Mrowka's space. Then taking $f : \psi^* \to S$ where f maps all the nonisolated points of ψ^* into a single point one obtains a closed mapping of a Moore space onto a convergent sequence which is not compact-covering[10].

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