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A Note on Spaces with Regular G_δ-Diagonals *

CHEN Huan-ran[1], *LIN Shou*[2]

1. Basic Courses Dept., Hunan Commercial College, Changsha 410215;
2. Math. Dept., Ningde Teachers' College, Fujian 352100

Abstract: In this short note, using a Mysior's example shows that a space with a regular G_δ-diagonal is not preserved by a finite-to-one and open map.

Key words: regular G_δ-diagonal; finite-to-one map; open map; submetrizable space.

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In this paper, all spaces are regular and T_1.N and R denote the sets of natural numbers and real numbers, respectively. In [1], Mysior gave a simple example of a non-realcompact space which is the union of two closed realcompact subspaces as follows.

Example 1 Let $X = R \times R$. All points (x, y) with $y \neq 0$ are assumed to be isolated. A base of neighborhoods of a point $(x, 0)$ is the family $\{U(x,n) : n \in N\}$ where each $U(x,n)$ is the union of three segments:

$$\{(x,y) : -1/n < y < 1/n\},$$
$$\{(x+1+y, y) : 0 < y < 1/n\},$$
$$\{(x+\sqrt{2}+y, -y) : 0 < y < 1/n\}.$$

Then X is a completely regular space which is an image of a metric space under a finite-to-one and open map[4].

A space (X, τ) is submetrizable if there exists a topology τ' on X such that $\tau' \subset \tau$ and (X, τ') is metrizable. A space X is said to have a (regular) G_δ-diagonal if the diagonal of X is a (regular) G_δ-set in X^2. A submetrizable space has a regular G_δ-diagonal. A space with a regular G_δ-diagonal has a G_δ-diagonal. We knew that a finite-to-one and open map preserves the property with a G_δ-diagonal[2]. It is a question whether submetrizability or the property with a regular G_δ-diagonal is preserved by a finite-to-one and open map. We give a negative answer to the above question by proving the space X in Example 1 without any regular G_δ-diagonal.

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In fact, if X has a regular G_δ-diagonal, by Theorem 1 in [3], there exists a sequence $\{\mu_n\}$ of open covers of X such that if x and y are distinct two points of X, then there are an integer n and open set U and V containing x and y respectively such that $U \cap \mathrm{st}(V, \mu_n) \neq \emptyset$. We can assume that every element of μ_n belongs to the base of neighborhood of X. Then for each $m \in N, r \in R$, there is an $m(r) \in N$ with $U(r, m(r)) \in \mu_m$. Let $R_k = \{r \in R : m(r) = k\}$.

Then $R = \cup_{k \in N} R_k$. Thus $\mathrm{int}_\tau(\mathrm{cl}_\tau(R_{\sigma(m)})) \neq \emptyset$ for some $\sigma(m) \in N$, where τ is the usual Euclidean topology on R. By induction principle, we can construct a sequence $\{R_{n,\sigma(n)}\}$ of subsets of R, here $\sigma : N \to N$, satisfying

(1) $U(r, \sigma(n)) \in \mu_n$ for each $r \in R_{n,\sigma(n)}$.

(2) $\emptyset \neq \mathrm{cl}_\tau(R_{n+1,\sigma(n+1)}) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(R_{n,\sigma(n)}))$ for each $n \in N$.

Take a point $a \in \cap_{n \in N} \mathrm{cl}_\tau((R_{n,\sigma(n)}))$. Put $x = (a - \sqrt{2}, 0), y = (a - 1, 0)$.

Then $U \cap \mathrm{st}(V, \mu_n) \neq \emptyset$ for each $n \in N$, and open sets U and V containing x and y respectively. In fact, for each $n, i, j \in N$, take an integer $k > \max\{\sigma(n), i, j\}$. Let $b \in (a, a + 1/k) \cap R_{n,\sigma(n)}$, then $(b, -(b - a)) \in U(x, i) \cap \mathrm{st}(U(y, j), \mu_n)$, a contradiction. Hence X has not any regular G_δ-diagonal.

Corollary *Submetrizability or property with a regular G_δ-diagonal is not preserved by a finite-to-one and open map.*

Suppose ρ is a family of subsets of a space X.ρ is a k-network for X, if K is a compact subset of X and U is an open neighborhood of K in X, there exists a finite subfamily ρ' of ρ with $K \subset \cup\rho' \subset U$. A space X is an $\aleph$-space if X has a σ-locally finite k-network. It is obvious that a metric space is an $\aleph$-space, and an $\aleph$-space has a G_δ-diagonal. It is a question posed by the second author of this paper in [4] whether every $\aleph$-space has a regualar G_δ-diagonal. We construct an example by revised Example 1 answering the above question negatively.

Example 2 There exists an $\aleph$-space without any regular G_δ-diagonal.

Take $Y = R \times R, P(Y)$ denotes the power set of Y. Let $S = \{F \in P(Y)^{R \times N} : x \in R, n \in N$, and $0 < y < 1/n$, there exist $a(y), b(y), c(y)$ and $d(y) \in R$ such that $a(y) < x, b(y) < x, c(y) < x + y + 1, d(y) < x + y + \sqrt{2}$ and

$$F(x,n) = \{(z,y) : 0 < y < 1/n, a(y) < z < x \text{ or } c(y) < z < x + y + 1\} \cup \{(z,-y) : 0 < y < 1/n, b(y) < z < x \text{ or } d(y) < z < x + y + \sqrt{2}\}.$$

For each $x \in R, n \in N$ and $F \in S$, put $V(x, n, F) = \{(x, 0)\} \cup F(x, n)$.

The set Y is endowed the following topology: all points (x, y) with $y \neq 0$ are assumed to be isolated; a base of neighborhoods of a point $(x, 0)$ is the family $\{V(x, n, F) : n \in N, F \in S\}$. Then Y is a completely regular space.

For each $n \in N$, define

$$Y_n = \begin{cases} R \times \{0\}, n = 1 \\ \{(x,y) : |y| \geq 1/n\}, n > 1. \end{cases}$$

Then Y_n is a closed discrete subspace of Y, and $Y = \cup_{n \in N} Y_n$. By the same way in Example in [5], all compact subspaces of Y is finite. Thus $\{\{y\} : y \in Y\}$ is a σ-locally finite k-network for Y, and Y is an $\aleph$-space. By the same method in Example 1, we can prove that Y has not any regular G_δ-diagonal.

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具有正则 G_δ 对角线空间的注记

陈 焕 然[1]， 林 寿[2]

1. 湖南商学院基础部，长沙 410215;

2. 宁德师专数学系，福建 352100

摘要: 这篇简短的注记使用 Mysior 的例子说明具有正则 G_δ 对角线的空间不被有限到一的开映射保持.