Questions and Answers in General Topology

Vol. 7, No. 1 1989

Publisher: Symposium of General Topology

Q & A in General Topology, Vol. 7 (1989)

PERFECT PREIMAGES OF SOME GENERALIZED METRIZABLE SPACES

Shou Lin

Department of Mathematics, Ningde Teachers' College, Ningde, Fujian, CHINA

All spaces are regular and T_1, and all mappings are continuous and onto.

In this paper we discuss the perfect preimages of generalized metrizable spaces having a G_δ-diagonal. The classes of spaces discussed are restricted to the one of spaces studied by Gruenhage in[6]. We knew that the perfect preimage of even a compact metric space need not have a G_δ-diagonal[9], hence for a class P of generalized metrizable spaces having a G_δ-diagonal, the following problem is considered:

PROBLEM. Suppose $f:X\to Y$ is a perfect mapping. If the space X has a G_δ-diagonal, and if the space Y belongs to the class P, does X belong to the class P?

The problem first was discussed by Okuyama and Borges. They independently proved that answer of the problem is positive for the class of metrizable spaces[2, 11]. Borges [2] still proved that answer of the problem is affirmative for the class of stratifiable spaces. Mancuso [9] proved that for the classes of $\aleph_0$-spaces, cosmic spaces or spaces with uniform bases, and Atkins and Slaughter[1] proved that for the classes of developable spaces, σ-spaces or semistratifiable spaces, answer of the problem is all true. In[8], we proved that for the class of $\aleph$-spaces, answer of the problem is still true. Moreover, Burke[3] constructed an example showing that answer of the problem is negative for the class of submetrizable spaces.

Theorem 1. Suppose there exists a perfect mapping f from the topological space X onto the semimetrizable space Y. If X has a G_δ-diagonal, then X is a semimetrizable space.

Proof. It is well-known that a space is a semimetrizable space if and only if it is a first countable semistratifiable space[4]. Since a perfect preimage of a semistratifiable space is a semistratifiable space if and only if it has a G_δ-diagonal[1], it is sufficient to show that X is first countable. Let $x \in X$, and let $y = f(x)$. Since Y is first countable, let (U_n) be a decreasing local base at y in Y. Since X has a G_δ-diagonal, and $f^{-1}(y)$ is a compact subspace of X, $f^{-1}(y)$ is metrizable. Let (V_n) be a decreasing local base at x in $f^{-1}(y)$. By regularity, choose an open set W_n in X such that $W_n \subset f^{-1}(U_n)$, $x \in W_n \cap f^{-1}(y) \subset V_n$ and $cl(W_{n+1}) \subset W_n$ for all n. If (W_n) is not a local base at x in X, then there exists a neighbourhood N of x in X such that $W_n - N \neq \emptyset$ for all n. Take $x_n \in W_n - N$. Then

$$\bigcap_{m \geq 1} cl\{x_n : n \geq m\} \subset \bigcap_{m \geq 1} cl(W_m - N) \subset \bigcap_{m \geq 1} (cl(W_m) - N)$$

$$= \bigcap_{m \geq 1} (f^{-1}(U_m) \cap W_m) - N = \{x\} - N = \emptyset ,$$

hence set $\{x_n : n=1,2,\ldots\}$ has not cluster point in X, so all its subset are closed in X. Take i such that $W_n \cap f^{-1}(y) \subset N$ for all $n>i$. Then $\{x_n ; n>i\}$ is closed in X, so $f(\{x_n : n>i\})$ is closed in Y. On the other hand, $x_n \in f^{-1}(U_n) - f^{-1}(y)$, thus $y \in cl(f(\{x_n : n>i\})) - f(\{x_n : n>i\})$, a contradiction. Therefore X is a first countable space, and X is a semimetrizable space.

A closed image of a metrizable space is called a Lašnev space . X is a Fréchet space if for every $A \subset X$ and every $x \in cl(A)$ there exists a sequence of points of A converging to x. Every Lašnev space is Fréchet. Let X be a non-metrizable Lašnev space , and let I be the unit interval with the usual topology. Then $X \times I$ is not a Lašnev space[10]. But the projective mapping $P: X \times I \to X$ is perfect, and $X \times I$ has a G_δ-diagonal, answer of the problem above is negative for the class of Lašnev spaces. Despite all this, we have the following result.

Theorem 2. Suppose there exists a perfect mapping f from the topological space X onto the Lašnev space Y. If X is a Fréchet space with a G_δ-diagonal, then X is a Lašnev space.

Proof. Since Y is a Lašnev space, Y is a paracompact space, and since f is perfect, X is a paracompact space. For the paracompact space X with a G_δ-diagonal, there exists a metrizable space M and a continuous, one-to-one mappimg g from X onto M [6, Corollary 2.9]. Now we define mapping h from X into $Y \times M$ such that $h(x) = ((f(x), g(x))$ for each

$x \in X$. Clearly, h is continuous and one-to-one. Since f is perfect, h is perfect[5, Theorem 3.7.9]. So h(X) is a Fréchet subspace of $Y \times M$. Since every Fréchet subspace of the product of countably many Lašnev spaces is a Lašnev space[12], h(X) is a Lašnev space, and since h is perfect and one-to-one, h is a homeomorphism embedding, hence X is Lašnev.

Corollary 3. Suppose Y is a Lašnev space and $f: X \to Y$ is an open, closed, finite-to-one mapping, then X is a Lašnev space.

Proof. Since Y is Lašnev, Y is a σ-space, and since f is open and finite-to-one, X is a σ-space[7]. Hence X has a G_δ-diagonal. For $x_1 \in cl(A) \subset X$, let $f^{-1}f(x_1) = \{x_1, x_2, \ldots, x_m\}$. There exists a collection $\{V_1, V_2, \ldots, V_m\}$ of open subsets of X such that $x_i \in V_i \subset X - cl(\bigcup\{V_j : i \neq j \leq m\})$ for each $i \leq m$. Since $x_1 \in cl(V_1 \cap A)$, $f(x_1) \in cl(f(V_1 \cap A))$. Then there exists a sequence $\{z_n\}$ in $V_1 \cap A$ such that $\{f(z_n)\}$ converges to $f(x_1)$. Since $K = \{f(x_1)\} \cup \{f(z_n) : \text{all } n\}$ is a compact subset of Y, $f^{-1}(K)$ is a compact subset of X. So $f^{-1}(K)$ is a compact space with a G_δ-diagonal, hence $f^{-1}(K)$ is a compact metrizable space. Therefore the sequence $\{z_n\}$ in $f^{-1}(K)$ has a convergent subsequence $\{z_{n_i}\}$. Suppose $\{z_{n_i}\}$ converges to $z \in X$. Then $z \in cl(V_1)$, and $f(x_1) = f(z)$, thus $x_1 = z$. Hence the sequence $\{z_{n_i}\}$ in A converges to x_1. X is a Fréchet space. Therefore X is a Lašnev space by Theorem 2 .

A space is a Lašnev space if and only if it is a Fréchet space with a σ-hereditarily closure-preserving k-network. From Theorem 2, one may conjecture that a space with a G_δ-diagonal has a σ-hereditarily closure-preserving k-network if it is a perfect preimage of a space with a σ-hereditarily closure-preserving k-network. It is not correct. In [13] we proved that for every non-$\aleph$ Lašnev space X, $X \times I$ has not a σ-hereditarily closure-preserving k-network, thus the space $X \times I$ with a G_δ-diagonal is a perfect preimage of the space X with a σ-hereditarily closure-preserving k-network.

Problem. Is a perfect preimage of an M_1-space an M_1-space if it has a G_δ-diagonal ?

References

1. J. Mr. Atkins and Slaughter, Jr., On the metrizability of preimages of metric spaces under the closed continuons functions, Proc. Oklahoma Top. Conf., 1972, 13-22. MR 54 # 13864.

2. C. J. R. Borges, On stratifiable spaces, Pacific J. Math., 17(1966),1-16.

3. D. K. Burke, On k-semimetric spaces, Proc. Amer. Math. Soc., 73(1979), 126-128.

4. G. Creede, Concerning semi-stratifiable spaces, Pacific J. Math., 22(1970), 47-54.

5. R. Engelking, General Topology, Polish Scientific Publ., Warszawa, 1977.

6. G. Gruenhage, Generalized metric spaces, Handbook of Set - Theoretic Topology, North-Holland, 1984, 423-502.

7. R. F. Gittings, Open mapping theory, Set-Theoretic Topology, Academic Press, 1977, 141-191.

8. S. Lin, Mapping theorems on $\aleph$-spaces, Top. Appl., (to appear).

9. V. J. Mancuso, Inverse images and first countability, Gen. Top. Appl., 2(1972), 29-44.

10. E. Michael, A quintuple quotient quest, Gen. Top, Appl., 2(1972), 91-138.

11. A. Okuyama, On metrizability of M-spaces, Proc. Japan Acad., 40(1964), 176-179.
12. K. Tamano, Closed images of metric spaces and metrization, Topology Proc., 10(1985), 177-186.
13. S. Lin, On a problem of K. Tamano, Q & A in Gen. Top., 6(1983), 99-102.

Received September 17, 1988