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On a Generalization of Michael's Theorem

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Abstract

A continuous map $f: X \to Y$ is a strong s-map if for each $y \in Y$, there exists a neighbourhood V of y in Y such that $f^{-1}(V)$ is a separable subspace of X. In this paper, it is shown that a regular space has a locally countable k-network if and only if it is a compact-covering, strong s-image of a metrizable space. This is a generalization of Michael's theorem: A regular space has a countable k-network if and only if it is a compact-covering image of a separable metrizable space. As a corollary, we prove that a regular k-space has a locally countable k-network if and only if it is a (compact-covering) quotient strong s-image of a metrizable space.

Key Words k-network; k-space; Compact-covering Map; Strong s-map

At Prage symposium on topology in 1961, Alexandroff suggested that by means of various maps the relationships between various classes of topological spaces are established. One of the successful results in this direction is Michael's theorem (cf.[1]): A regular space has a countable k-network if and only if it is a compact-covering image of a separable metrizable space. Which class of topological space may Michael's theorem be generalized to? Since there exists a locally compact metric space M and a compact-covering, quotient, finite-to-one map $f: M \to X$ such that a regular space X is not a space with a σ-locally countable k-network (Example), and since every metric space has a σ-locally finite base, it is necessary that an appropriate condition should be added to the map in consideration. In this paper, we introduce the concept of strong s-map, and prove that a regular space has a locally countable k-network if and only if it is a compact-covering strong s-image of a metrizable space (Theorem). This theorem is a generalization of Michael's theorem because a continuous map defined on a separable space is a strong s-map.

In the following, all maps are continuous and surjective, and N denotes the set of positive integers.

Let X be a topological space. A collection $\mathscr{P}$ of subsets of X is called a k-network if for any compact subset of an open set U of X, there exists a fi-

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nite sub-collection $\mathscr{P}'$ of $\mathscr{P}$ such that $K \subset \cup\mathscr{P}' \subset U$. A map $f: X \to Y$ is called compact-covering if every compact subset of Y is the image of a compact subset of X. f is called a strong s-map, or an ss-map for short, if for each $y \in Y$, there exists a neighbourhood V of y in Y such that $f^{-1}(V)$ is a separable subspace of X.

Theorem A regular space has a locally countable k-network if and only if it is a compact-covering ss-image of a metrizable space.

Proof. Suppose a space X is an image of a metrizable space M under a compact-covering ss-map f, i.e., $f: M \to X$. By the classical Nagata-Smirnov metrization theorem, M has a σ-locally finite base $\mathscr{B}$. Since a base is a k-network and a k-network is preserved under compact-covering maps, $\mathscr{P} = \{f(B): B \in \mathscr{B}\}$ is a k-network for X. For each $x \in X$, since f is an ss-map, there exists a neighbourhood V of x in X such that $f^{-1}(V)$ is a separable subspace of M. $f^{-1}(V)$ is a Lindelöf subspace of M because M is a metric space. Since a locally finite family of Lindelöf spaces is countable, $\{B \in \mathscr{B}: B \cap f^{-1}(V) \neq \phi\}$ is countable. So $\{P \in \mathscr{P}: P \cap V \neq \phi\}$ is countable. Hence $\mathscr{P}$ is a locally countable k-network for X.

Suppose X is a regular space with a locally countable k-network. By regularity, we can assume that X has a locally countable closed k-network $\mathscr{P} = \{P_a: a \in A\}$ such that $P_a \neq P_b$ when $a \neq b$, which is closed under finite intersections. For each $i \in N$, let A_i denotes the set A with discrete topology. Put

$$M = \{b = (a_i) \in \prod_{i \in N} A_i: \text{ there exists } \{P_{a_i}: i \in N\} \subset \mathscr{P},$$
$$\text{which forms a descending net at some } x_b \in X\},$$

and give M the subspace topology induced from the usual product topology of the discrete spaces A_i. x_b is unique in M because X is a Hausdorff space. We define $f: M \to X$ by $f(b) = x_b$ for each $b = (a_i) \in M$, i.e., $f((a_i)) = \bigcap_{i \in N} P_{a_i} = x_b$. We will show that f is a compact-covering ss-map from M onto X.

1. f is a continuous and surjective map: Since $\mathscr{P}$ is a point countable k-network which is closed under finite intersections for X, it is easy to check that f is surjective. Let $b = (a_i) \in M$, and let $x_b = f(b)$. Suppose V is an open neighbourhood of x_b in X. Since $\{P_{a_i}: i \in N\}$ forms a descending net at x_b, there exists $n \in N$ such that $x_b \in P_{a_i} \subset V$. Put $W = \{c \in M:$ the n-th coordinate of c is $a_n\}$. Then W is an open neighbourhood of b in M, and $f(W) \subset P_{a_n} \subset V$. Hence f is continuous.

2. f is an ss-map: For each $x \in X$, since $\mathscr{P}$ is locally countable, there exists a neighbourhood V of x in X such that $V \cap P \neq \phi$ for only countable many

$P\in\mathscr{P}$. So $\left(\prod_{i\in N}\{a\in A_i: V\cap P_a\neq\phi\}\right)\cap M$ is a separable subspace of M, and

$$f^{-1}(V)\subset\left(\prod_{i\in N}\{a\in A_i: V\cap P_a\neq\phi\}\right)\cap M.$$

Hence f is an ss-map.

3. f is a compact-covering map: Suppose K is a non-empty compact subset of X. Since $\mathscr{P}$ is locally countable, $\mathscr{P}(K)=\{P\in\mathscr{P}: K\cap P\neq\phi\}$ is countable. So

$$\mathscr{P}'(K)=\{\mathscr{F}\subset\mathscr{P}(K): \mathscr{F}\text{ is finite and }K\subset U\mathscr{F}\}$$

is countable. Let

$$\mathscr{P}'(K)=\{\mathscr{P}'_i: i\in N\}.$$

For each $n\in N$, put

$$\mathscr{P}_n=\left\{\bigcap_{i\leqslant n}P_i: P_i\in\mathscr{P}'_i\text{ for each }i\leqslant n\right\}.$$

Then $\{\mathscr{P}_n\}_{n\in N}$ is a sequence of finite covers of K. Since $\mathscr{P}$ is closed under finite intersections, we can choose a finite subset A'_n of A_n such that

$$\mathscr{P}_n=\{P_a: a\in A'_n\}.$$

Let

$$L=\left\{b=(a_i)\in\prod_{i\in N}A'_i: P_{a_{i+1}}\subset P_{a_i}\text{ and }P_{a_i}\cap K\neq\phi\text{ for each }i\in N\right\}.$$

Since each A_i is a discrete space, and since each A'_i is finite, L is a compact subset of $\prod_{i\in N}A'_i$. If $b=(a_i)\in L$, then

$$K\cap\left(\bigcap_{i\in N}P_{a_i}\right)\neq\phi.$$

Take $x\in K\cap\left(\bigcap_{i\in N}P_{a_i}\right)$. We will show that $\{P_{a_i}\}_{i\in N}$ forms a net at x for X.

In fact, for an open neighbourhood V of x in X, we can find an open neighbourhood W of x in K such that $x\in \mathrm{cl}_X(W)\subset V$. Since $\mathrm{cl}_X(W)$ is a compact subset of X, there exists a finite subcollection $\mathscr{P}'$ of $\mathscr{P}$ such that $\mathrm{cl}_X(W)\subset U\mathscr{P}'\subset V$. Take a finite sub-collection $\mathscr{P}''$ of $\mathscr{P}$ such that $K-W\subset U\mathscr{P}''\subset X-\{x\}$. Then $K\subset U(\mathscr{P}'\cup\mathscr{P}'')$. So there exists $n\in N$ such that $\mathscr{P}'_n\subset\mathscr{P}'\cup\mathscr{P}''$. Since $x\in P_{a_n}\in\mathscr{P}_n$, there exists $P\in\mathscr{P}'\cup\mathscr{P}''$ such that $x\in P_{a_n}\subset P$, and $P\in\mathscr{P}'$. Hence $x\in P_{a_n}\subset V$, and $\{P_{a_i}\}_{i\in N}$ forms a net at x for X. By the definition of the space M, $b\in M$ and $f(b)=x\in K$. Therefore, $L\subset M$ and $f(L)\subset K$. On the other hand, if $x\in K$, by the property of $\{\mathscr{P}_n\}_{n\in N}$, there exists $P_{a_n}\in\mathscr{P}_n$ such that $x\in P_{a_{n+1}}\subset P_{a_n}$, where $a_n\in A'_n$ for each $n\in N$. Let $b=(a_n)$. Then $b\in L$ and $f(b)=x$

(because $x \in K \cap \left(\bigcap_{n \in N} P_{\alpha_n} \right)$, and $\{P_{\alpha_n}\}_{n \in N}$ forms a descending net at x for X).

So $f(L) = K$, and f is a compact-covering map.

Corollary 1[1] A regular space has a countable k-network if and only if it is a compact-covering image of a separable metric space.

Proof. Let $f: M \to X$ be a compact-covering map, where M is a separable metric space. Since M has a countable base, and since a k-network is preserved under a compact-covering map, X has a countable k-network.

Suppose a regular space X has a countable k-network. By Theorem, there exists a compact-covering ss-map f from M onto X, where M is a metrizable space. Since X is a Lindelöf space (cf. [1], (D)), and since f is an ss-map, there exists a countable open cover $\{V_i: i \in N\}$ of X such that each $f^{-1}(V_i)$ is a separable subspace of M. Thus,

$$M = \bigcup_{i \in N} f^{-1}(V_i)$$

is separable, and it is a separable metric space.

Remark Let M be a locally separable metrizable space which is not a separable space. Then it has a locally countable k-network (in fact, it has a locally countable base), and has no countable k-network (because a space with a countable k-network is separable). Hence Theorem is a real improvement of Michael's theorem.

Corollary 2 The following are equivalent for a regular space X:

(1) X is a k-space with a locally countable k-network.

(2) X is a compact-covering quotient ss-image of a metrizable space.

(3) X is a quotient ss-image of a metrizable space.

Proof. If an image of a compact-covering map is a Hausdorff k-space, then the map is quotient (cf. [2], Lemma 45.8). By Theorem, (1)⇒(2) holds. (2)⇒(3) is obvious. Finally, suppose $f: M \to X$ is a quotient ss-map, where M is a metrizable space. By the proof of sufficiency of Theorem, there exists a locally countable base $\mathscr{B}$ of M such that

$$\mathscr{P} = \{f(B): B \in \mathscr{B}\}$$

is a locally countable family of subsets of X. We will show that $\mathscr{P}$ is a k-network for X. if K is a compact subset of an open set V of X, then

$$\mathscr{P}' = \{P \in \mathscr{P}: P \cap K \neq \phi \text{ and } P \subset V\}$$

is countable, because a locally countable family of subsets of a compact space is countable. Let

$$\mathscr{P}' = \{P_i: i \in N\},$$

If for each $i\in N$, $K\not\subset\bigcup_{j\le i}P_j$, then there exists $x_i\in K-\bigcup_{j\le i}P_j$. Since a net is preserved under continuous maps, $\{P_i\cap K: i\in N\}$ is a countable net for the compact space K. So K is a metrizable space (cf. [1], §10). Hence the sequence $\{X_i\}$ in K has a convergent subsequence. Without loss of generality, we can assume that the sequence $\{x_i\}$ itself converges to $x\in K$, and no x_i is equal to x. Thus $X'=\{x_i: i\in N\}$ is not closed in X. Since f is quotient, $f^{-1}(X')$ is not closed in M. Take

$$y\in \mathrm{cl}(f^{-1}(X'))-f^{-1}(X').$$

Then $y\in f^{-1}(K)\subset f^{-1}(V)$ because

$$\mathrm{cl}(f^{-1}(X'))\subset f^{-1}(\mathrm{cl}(X'))\subset f^{-1}(K).$$

Since $\mathscr{B}$ is a base for M, there exists $B\in\mathscr{B}$ such that

$$y\in B\subset f^{-1}(V),$$

and

$$f^{-1}(x_i)\cap B\neq\phi$$

for infinitely many $i\in N$. Hence $f(B)\in\mathscr{P}'$ and $x_i\in f(B)$ for infinitely many $i\in N$. This contradicts the selection of x_i. Therefore, there exists $i\in N$ such that

$$K\subset\bigcup_{j\le i}P_j\subset V,$$

and $\mathscr{P}$ is a locally countable k-network for X. Since M is a k-space, and since k-space property is preserved under quotient maps, X is a k-space. This completes the proof of the Corollary.

Let $f: M\to X$ be an ss-map. Then M is a locally separable space. If M is a metrizable space, f is an s-map (i.e., $f^{-1}(x)$ is a separable subspace of M for each $x\in X$). We will construct an example to show that the condition "f is an ss-map" in Theorem and Corollary 2 can not be replaced by the condition "f is an s-map, and $f^{-1}(X)$ is locally separable".

Example There exists a locally compact metrizable space M and a compact-covering, quotient, two-to-one map f from M onto X such that X is a regular separable space which has not any σ-locally countable k-network.

In fact, we will show that Example 9.3 in [3] satisfies all requirements mentioned above. Let

$$S=\left\{\frac{1}{n}: n\in N\right\}\cup\{0\},\qquad X=[0,1]\times S.$$

And let

$$Y=[0,1]\times\{1/n: n\in N\}$$

have the usual Euclidean topology as a subspace of $[0,1]\times S$. Define a typical neighbourhood of $(t,0)$ in X to be of the form

$$\{(t,0)\}\cup\left(\bigcup_{k\geq n}V(t,\ 1/k)\right),$$

where $V(t,\ 1/k)$ is a neighbourhood of $(t,\ 1/k)$ in $[0,1]\times\{1/k\}$. Put

$$M=(\bigoplus_{n\in N}[0,1]\times\{1/n\})\oplus(\bigoplus_{t\in[0,1]}\{t\}\times S),$$

and define f from M onto X such that f is an obvious map. Then f is a quotient two-to-one map from the locally compact metrizable space M onto separable, regular, non-meta-Lindelöf, k-space X (cf. [3]). f is a compact-covering map. In fact, let K be a compact subset of X. Since $[0,1]\times\{0\}$ is a closed discrete subspace of X, $K\cap([0,1]\times\{0\})$ is finite, which we assume to be $\{(t_i,0):i\leq m\}$. Let

$$K_0=\bigcup_{i\leq m}(\{t_i\}\times S).$$

If for each $n\in N$,

$$K\not\subset K_0\cup\left(\bigcup_{j\leq n}[0,1]\times\{1/j\}\right),$$

i.e.,

$$K-K_0\cup\left(\bigcup_{j\leq n}[0,1]\times\{1/j\}\right)\neq\phi,$$

then there exists a subsequence $\{j_n\}$ of $\{n\}$ and a subset $\{x_n:n\in N\}$ of X such that

$$x_n\in K\cap([0,1]\times\{1/j_n\})-K_0$$

for each $n\in N$. Since X is the union of countably many closed metrizable subspaces, X is a σ-space. So the compact subspace K of X is a compact metrizable subspace. Thus, the sequence $\{x_n\}$ has a convergent subsequence. We can assume that the sequence $\{x_n\}$ itself converges to x. Since $j_n\to+\infty$, $x\in K\cap([0,1]\times\{0\})$, i.e., there exists $i\leq m$ with $x=(t_i,0)$. And since $x_n\in[0,1]\times\{1/j_n\}-K_0$, there exists an open neighbourhood $V(t_i,1/j_n)$ of $(t_i,1/j_n)$ in $[0,1]\times\{1/j_n\}$ such that

$$x_n\in[0,1]\times\{1/j_n\}-V(t_i,1/j_n).$$

For each $k\in N-\{j_n:n\in N\}$, let

$$V(t_i,1/k)=[0,1]\times\{1/k\}.$$

Then

$$W=\{x\}\cup\left(\bigcup_{k\in N}V(t_i,1/k)\right)$$

is an open neighbourhood of x in X such that all $x_n\notin W$, a contradiction. So there exists $n\in N$ such that

$$K \subset \left(\bigcup_{j \leqslant n} [0,1] \times \{1/j\} \right) \cup K_0.$$

Put

$$L = (\bigoplus_{j \leqslant n} ([0,1] \times \{1/j\}) \cap K) \oplus (\bigoplus_{i \leqslant m} (\{t_i\} \times S) \cap K).$$

Then L is a compact subset of M, and $f(L) = M$. Hence f is a compact-covering map. Finally, X has not any σ-locally countable k-network, since a regular sequential space with a σ-locally countable k-network is a meta-Lindelöf space (cf.[4], Proposition 1), and since a regular k-space with point G_δ-property is a sequential space (cf.[5], Theorem 7.3).

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