SPACES HAVING σ-HEREDITARILY CLOSURE-PRESERVING k-NETWORKS

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ABSTRACT. In this paper, it is shown that a regular k-space is a cs-σ-space if it has a σ-hereditarily closure-preserving cs-network.

Throughout, all spaces are assumed to be regular and Hausdorff.

k-networks and cs-networks are two distinct concepts, but under a certain condition they are equivalent. For example, a space with a σ-locally finite k-network is equivalent to the space with a σ-locally finite cs-network [2]. As is well known, a Lašnev space has a σ-hereditarily closure-preserving k-network. A problem is whether a Lašnev space has a σ-hereditarily closure-preserving cs-network. In the first part we give a negative answer to this problem. Next, we knew that a space having a σ-hereditarily closure-preserving base is equivalent to the space having a σ-locally finite base [1], and a natural problem is whether a space having a σ-hereditarily closure-preserving cs-network is a space having a σ-locally finite cs-network. In the second part we give a partial answer to this problem. Finally, we knew that a Fréchet space having a σ-hereditarily closure-preserving k-network is the closed image of a metric space [3], and an interesting and important problem is whether a space having a σ-hereditarily clousre-preserving k-network is a closed image of an $\aleph$-space. If its answer is affirmative, then every space having a σ-hereditarily closure-preserving k-network is the union of an $\aleph$-space and a σ-closed discrete space. In the third part we show that the last conclusion is true. It improves a result in [9].

First of all, let us recall some definitions. Let $\mathcal{P}$ be a family of subsets of a space X. $\mathcal{P}$ is hereditarily closure-preserving (abbrev. HCP) if, whenever a subset $S(P) \subset P$ is chosen for each $P \in \mathcal{P}$, the resulting family $\{S(P) : P \in \mathcal{P}\}$ is closure-preserving. $\mathcal{P}$ is a k-network for X if whenever K is a compact set of an open set U of X, then $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. $\mathcal{P}$ is a cs-network for X if for every convergent sequence Z in X and neighborhood U of Z, there is a $P \in \mathcal{P}$ such that Z is eventually in P and $P \subset U$. A space X is an $\aleph$-space (resp. cs-σ-space) if X has a σ-locally finite k-network (resp. cs-network). An $\aleph$-space is equivalent to a cs-σ-space [2].

Recall that the space S_{ω_1} is the quotient space obtained from the topological sum of ω_1 non-trivial convergent sequences by identifying all the limit points. S_{ω_1} has a σ-HCP k-network because it is a Lašnev space [3].

Theorem 1. *S_{ω_1} has not σ-HCP cs-networks.*

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Proof. Let $X = \cup\{X_\alpha : \alpha < \omega_1\}$, where $X_\alpha = \{0\} \cup \{x_{\alpha,n} : n \in N\}$ with each $x_{\alpha,n}$ is an isolated point in X, and U is an open neighborhood of the point 0 in X if and only if for each $\alpha < \omega_1$, there exists $m(\alpha) \in N$ such that

$$\cup\{\{0\} \cup \{x_{\alpha,n} : n \geq m(\alpha)\} : \alpha < \omega_1\} \subset U,$$

then X is homeomorphic to S_{ω_1}.

If X has a σ-HCP cs-network. Let $\cup\{\mathcal{P}_n : n \in N\}$ be a σ-HCP cs-network for X, where each $\mathcal{P}_n$ is HCP. For each $\alpha < \omega_1, n \in N$, put

$$k_{\alpha,2n-1} = x_{1,n}$$

$$k_{\alpha,2n} = x_{\alpha,n}$$

$$K_\alpha = \{0\} \cup \{k_{\alpha,n} : n \in N\}$$

$$K_{\alpha,n} = \{0\} \cup \{k_{\alpha,m} : m \geq n\}.$$

By transfinite induction, for each $\alpha < \omega_1$, there are $n(\alpha), m(\alpha) \in N$ such that for some $P_\alpha \in \mathcal{P}_{n(\alpha)}$,

$$K_{1,m(1)} \subset P_1 \subset X, K_{\alpha,m(\alpha)} \subset P_\alpha \subset X - \{x_\beta : \beta < \alpha\}, \alpha > 1$$

where $x_\beta \in X_\beta \cap K_{\beta,m(\beta)} - \{0\}$. For each $k \in N$, put

$$A_k = \{\alpha < \omega_1 : n(\alpha) = k\},$$

then there is a $k \in N$ with $|A_k| = \aleph_1$. Let $\mathcal{P} = \{P_\alpha : \alpha \in A_k\}$, then $\mathcal{P} \subset \mathcal{P}_k$. For each $\alpha, \beta \in A_k$, if $\beta < \alpha$, then

$$x_\beta \in P_\beta, \quad \text{and} \quad x_\alpha \in P_\alpha \subset X - \{x_\beta\},$$

hence $P_\alpha \neq P_\beta$, and $\mathcal{P}$ is HCP. For $\alpha \in A_k$, there exists $h(\alpha) \in N$ with $X_1 \cap K_{\alpha,m(\alpha)} = \{0\} \cup \{x_{1,n} : n \geq h(\alpha)\}$. For each $i \in N$, take $\alpha_i \in A_k, n_i \in N$ with

$$\alpha_i < \alpha_{i+1}, h(\alpha_i) < n_i < n_{i+1}.$$

Then $x_{1,n_i} \in P_{\alpha_i} \in \mathcal{P}$, and $\{x_{1,n_i} : i \in N\}$ is a discrete closed subspace of X. This contradicts with sequence $\{x_{1,n_i}\}$ converging to point 0. Therefore S_{ω_1} has not σ-HCP cs-networks.

By Theorem 1, a space with a σ-HCP k-network can not be a space with a σ-HCP cs-network. But, its converse proposition is true by the next Theorem 2. First, we establish a Lemma which is proved in [4].

Lemma 1. *If $\mathcal{P}$ is an HCP collection of X, then $\{\bar{P} : P \in \mathcal{P}\}$ also is an HCP collection of X.*

Theorem 2. *If a space X has a σ-HCP cs-network, then X has a σ-HCP k-network.*

Proof. Let $\cup\{\mathcal{P}_n : n \in N\}$ be a σ-HCP cs-network for X, where each $\mathcal{P}_n$ in HCP. We can assume that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ and each element of $\mathcal{P}_n$ is a closed subset of X by Lemma 1. For a compact set K of an open set U of X, put

$$\mathcal{P}'_n = \{P \in \mathcal{P}_n : Z \subset P \subset U \text{ for some convergent sequence } Z\}$$

$F_n = \cup\mathcal{P}'_n$,

then $K \subset F_n$ for some $n \in N$. Otherwise, there exists $x_n \in K - F_n$ for each $n \in N$. Thus $\{x_n : n \in N\}$ is an infinite subset of K, and there is a subsequence $\{x_{n_i}\}$ which converges to some point $x \in K$ because K is a compact metric subspace. Hence $\{x\} \cup \{x_{n_i} : i \geq m\} \subset P \subset U$ for some $m \in N$ and $P \in \mathcal{P}_n$, so $P \subset F_n$. Take $i \in N$ with $n_i \geq \max\{n, m\}$, then $x_{n_i} \in (K - F_{n_i}) \cap P = \emptyset$, a contradiction. So for some $n \in N, K \subset F_n$, i.e., $\mathcal{P}'_n$ is an HCP closed cover of the compact subset K of X, thus there exists a finite subfamily $\mathcal{P}''_n$ of $\mathcal{P}'_n$ such that $\mathcal{P}''_n$ convers K[7], and $K \subset \cup\mathcal{P}''_n \subset U$. Hence $\cup\{\mathcal{P}_n : n \in N\}$ also is a σ-HCP k-network for X.

By Theorem 1 and Theorem 2, it is of interest to know whether a space with a σ-HCP cs-network is a cs-σ-space. One partial answer is given by the next Theorem 3.

Let $\mathcal{P}$ be a collection of subsets of X, put

$D(\mathcal{P}) = \{x \in X : \mathcal{P} \text{ is not point-finite at } x\}$,

$\mathcal{F}(\mathcal{P}) = \{\overline{P - D(\mathcal{P})} : P \in \mathcal{P}\} \cup \{\{x\} : x \in D(\mathcal{P})\}$,

then the following Lemma 2 is proved by Y. Tanaka in [8].

Lemma 2. *Suppose X is a k-space. If $\mathcal{P}$ is an HCP collection of closed subsets of X, then $D(\mathcal{P})$ is a discrete closed subspace of X.*

Lemma 3. *Suppose X is a sequential space. If $\mathcal{P}$ is an HCP collection of closed subsets of X, and any closed subspace of X is not homeomorphic to S_{ω_1}, then $\mathcal{F}(\mathcal{P})$ is point-countable.*

Proof. If $\mathcal{F}(\mathcal{P})$ is not a point-countable collection of X, then $x \in \cap\{\overline{P - D(\mathcal{P})} : P \in \mathcal{P}'\}$ for some $x \in X$ and some uncountable $\mathcal{P}' \subset \mathcal{P}$, so $x \in \cap\mathcal{P}'$, and $x \in D(\mathcal{P})$. Hence $x \in \overline{P - \{x\}}$ for each $P \in \mathcal{P}'$. Thus $P - \{x\}$ is not a closed subset of X. Since P is a closed subset of sequential space X, there exists a sequence $\{y_n\}$ consisting of distinct points of $P - \{x\}$ such that $\{y_n\}$ converges to x. But $(\{x\} \cup \{y_n : n \in N\}) \cap D(\mathcal{P})$ is finite because $\{x\} \cup \{y_n : n \in N\}$ is compact and $D(\mathcal{P})$ is a discrete closed subspace of X by Lemma 2, thus $\{y_n : n \in N\} - D(\mathcal{P})$ is infinite. Let

$$\{y_n : n \in N\} - D(\mathcal{P}) = \{x_n : n \in N\},$$

then sequence $\{x_n\}$ converges to x.

Now, for each $P \in \mathcal{P}'$, there exists a sequence $\{x(P, n)\}$ consisting of distinct points of $P - D(\mathcal{P})$ such that $\{x(P, n)\}$ converges to x. $\mathcal{P}'$ is point-finite at $x(P, n)$ because of $x(P, n) \bar{\in} D(\mathcal{P})$, thus for any countable subfamily $\mathcal{P}''$ of $\mathcal{P}'$,

$$\mathcal{P}' - \{Q \in \mathcal{P}' : x(P, n) \in Q, P \in \mathcal{P}'' \text{ and } n \in N\}$$

is still uncountable. By transfinite induction, take a subfamily $\{P_\alpha : \alpha < \omega_1\}$ of $\mathcal{P}'$ such that $x(P_{\alpha,n}) \neq x(P_{\beta,m})$ when $\alpha \neq \beta$. Put

$$Y = \{x\} \cup \{x(P_{\alpha,n}) : \alpha < \omega_1, n \in N\},$$

then Y is a closed subspace of X because $\mathcal{P}$ is HCP and $\{x\} \cup \{x(P_\alpha, n) : n \in N\} \subset P_\alpha$ for each $\alpha < \omega_1$. Obviously, Y is homeomorphic to S_{ω_1}, a contradiction.

Theorem 3. *If a k-space X has a σ-HCP cs-network, then X is a cs-σ-space.*

Proof. Since X has a σ-HCP *cs*-network, X is a σ-space. Thus each point of X is a G_δ-set of X, and X is a sequential space [6]. By Lemma 1 and Theorem 2, X has a σ-HCP closed k-network. Let $\cup\{\mathcal{P}_n : n \in N\}$ be a σ-HCP closed k-network, where each $\mathcal{P}_n$ is an HCP family of closed subsets of X. We can assume that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. Put

$$\mathcal{F} = \cup\{\mathcal{F}(\mathcal{P}_n) : n \in N\}.$$

If K is a compact subset and U is an open subset of X with $K \subset U$, then $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}_n$. Put

$$\mathcal{F}' = \{\overline{P - D(\mathcal{P}_n)} : P \in \mathcal{P}'\} \cup \{\{x\} : x \in K \cap D(\mathcal{P}_n)\},$$

then $\mathcal{F}'$ is a finite subfamily of $\mathcal{F}$ because $K \cap D(\mathcal{P}_n)$ is finite by Lemma 2, and $K \subset \cup\mathcal{F}' \subset U$. By Theorem 1, Lemma 2 and Lemma 3, $\mathcal{F}$ is a σ-closure-preserving and point-countable closed k-network. Hence X is an $\aleph$-space by the proof of Theorem 2.2 in [5], and X is a *cs*-σ-space. This completes the proof of Theorem 3.

Corollary. *If X is a Lašnev space, then X is an $\aleph$-space if and only if X has a σ-HCP cs-network.*

The last theorem in this paper is on the decomposition of spaces having a σ-HCH k-network.

Theorem 4. *If X has a σ-HCP k-network, then exists an $\aleph$-space Z of X such that $X - Z$ is a σ-closed discrete subspace of X.*

Proof. Let $\cup\{\mathcal{P}_n : n \in N\}$ be a σ-HCP k-network of X. We can assume that each $\mathcal{P}_n$ is an HCP family of closed subsets of X and $X \in \mathcal{P}_n \subset \mathcal{P}_{n+1}$. For each $n \in N, x \in X$, put

$$\mathcal{P}_{n,x} = \{P \in \mathcal{P}_n : x \in P\},$$

$$Y_n = \{x \in X : \cap\mathcal{P}_{n,x} = \{x\}\},$$

then Y_n is a closed discrete subspace of X by Lemma 2.5 in [7]. Put

$$Z = X - \cup\{Y_n : n \in N\},$$

then $\mathcal{P}_{n,x}$ is finite for each $x \in Z$. In fact, suppose $\mathcal{P}_{n,x}$ is infinite. For each $m \in N, \mathcal{P}_{m,x} - \{x\} \neq \emptyset$ because $x \bar{\in} Y_m$. By inductive method we can choose a subset $\{x_i : i \geq n\}$ of X and a subfamily $\{P_i : i \geq n\}$ of $\mathcal{P}_n$ such that $x_i \in P_i \cap (\cap\mathcal{P}_{i,x}) - \{x\}$. Let

$$V = X - \{x_i : i \geq n\},$$

then V is an open neighborhood of x in X, hence

$$x \in P \subset V \subset X - \{x_m\}$$

for some $m \geq n$ and $P \in \mathcal{P}_m$, a contradiction with $x_m \in \cap\mathcal{P}_{m,x}$. Thus each $\mathcal{P}_{n,x}$ is finite for $x \in Z, n \in N$. Put

$$\mathcal{F}_n = \{P \cap Z : P \in \mathcal{P}_n\},$$

then $\cup\{\mathcal{F}_n : n \in N\}$ is a σ-locally finite k-network of Z. Therefore Z is an $\aleph$-space.

REFERENCES

1. D. Burke, R. Engelking and D. Lutzer, *Hereditarily closure-preserving collections and metrization*, Proc. AMS., **51** (1975), 483–488.
2. L. Foged, *Characterizations of* $\aleph$*-spaces*, Pacific J. Math., **110** (1984), 59–63.
3. L. Foged, *A characterization of closed images of metric spaces*, Proc. AMS., **95** (1985), 487–490.
4. S. Lin, *On a problem of K. Tamano*, Questions and Answers in Gen. Top., **6** (1988), 99–102.
5. S. Lin, *Mapping theorems on* $\aleph$*-spaces*, Top. Appl., **30** (1988), 159–164.
6. E. Michael, *A quintuple quotient quest*, Gen. Top. Appl., **2** (1972), 71–138.
7. A. Okuyama, *On a generalization of* Σ*-spaces*, Pacific J. Math., **42** (1972), 485–495.
8. Y. Tanaka, *Decompositions of spaces determined by compact subsets*, Proc. AMS., **97** (1986), 549–555.
9. Y. Tanaka, *Point-countable covers and k-networks*, Top. Proc., **12** (1987), 327–349.

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