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ON A PROBLEM OF K. TAMANO

Shou Lin

Department of Mathematics, Ningde Teachers' College,

Ningde, Fujian, CHINA

ABSTRACT. In this paper, K. Tamano's problem of whether every subspace of the countably many products of Lašnev spaces has a σ-hereditarily closure-preserving k-network is answered negatively by proving that for a Lašnev space X, $X \times I$ has a σ-hereditarily closure-preserving k-network if and only if X has a σ-locally finite k-network.

It is known that the product of two Lašnev spaces need not be a Lašnev space because it need not be a Fréchet space. In [3], we proved that every Fréchet subspace of the countably many product of Lašnev spaces is a Lašnev space. Foged [1] proved that a space is a Lašnev space if and only if it is a Fréchet Hausdorff space with a σ-hereditarily closure-preserving k-network. The following problem was posed by K. Tamano in [5].

PROBLEM. Does every subspace of the countably many product of Lašnev spaces have a σ-hereditarily closure-preserving k-network ?

In this paper, we answer this problem negatively. All spaces are T_1. I denotes the unit interval with the usual topology. A closed image of a metric space is a Lašnev space. Lašnev spaces are paracompact. Let X be a topological space. A collection P of subsets of X is hereditarily closure-preserving if whenever a subset $C(P) \subset P$ is chosen for each $P \in P$, the resulting collection $\{C(P) : P \in P\}$ is colsure-preserving. A collection P of subsets of X is k-network for X if whenever K is a compact subset of an open set U of X, then $K \subset \cup P' \subset U$ for some finite subcollection P' of P.

Lemma 1. If P is a hereditarily closure-preserving collection for a regular space X, then $\{cl(P) : P \in P\}$ is hereditarily closure-preserving.

Proof. Let $P = \{P_a : a \in A\}$. If a subset $H_a \subset cl(P_a)$ is chosen for each $a \in A$ such that

$x \in cl(\cup\{H_a : a \in A\}) - \cup\{cl(H_a) : a \in A\}$ for some $x \in X$,

then for each $a \in A$, there exist open sets U_a, V_a of X such that $x \in U_a$, $cl(H_a) \subset V_a$, and $U_a \cap V_a = \phi$. Thus

$H_a \subset cl(P_a) \cap V_a \subset cl(P_a \cap V_a)$, and $x \in cl(\cup\{cl(P_a \cap V_a) : a \in A\}) = \cup\{cl(P_a \cap V_a) : a \in A\}$. Hence $x \in cl(P_a \cap V_a)$ for some $a \in A$. Therefore $U_a \cap (P_a \cap V_a) \neq \phi$, a contradiction.

The following Lemma can be obtained by using the techniques invented by A. Okuyama [4] (Theorem 3.7).

Lemma 2. Let X be a paracompact space, and let $\mathbf{P}$ be a σ-hereditarily closure-preserving family of closed subsets of $X \times I$. Then there exists a closed subspace Y of $X \times I$, and a perfect map f from Y onto X such that $\{f(P \cap Y) : P \in \mathbf{P}\}$ is a σ-locally finite family of subsets of X.

Theorem. Let X be a Lašnev space. If $X \times I$ has a σ-hereditarily closure-preserving k-network, then X has a σ-locally finite k-network.

Proof. Let $\mathbf{F}$ be a σ-hereditarily closure-preserving k-network for $X \times I$. By the regularity of $X \times I$ and Lemma 1, $\mathbf{F} = \{cl(F) : F \in \mathbf{F}\}$ is a σ-hereditarily closure-preserving closed k-network for $X \times I$. By Lemma 2, there exists a closed subspace Y of $X \times I$, and a perfect map f from Y onto X such that $\mathbf{M} = \{f(P \cap Y) : P \in \mathbf{P}\}$ is a σ-locally finite family of subsets of X. Since $\{P \cap Y : P \in \mathbf{P}\}$ is a k-network for Y, $\mathbf{M}$ is a σ-locally finite k-network for X.

Since a Lašnev space need not have a σ-locally finite k-network [2, example 2.6], the Theorem answers negatively above question.

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