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Strict Pytkeev networks with sensors and their applications in topological groups [☆]

Xin Liu [a], Chuan Liu [b], Shou Lin [a,*]

[a] Institute of Mathematics, Ningde Normal University, Ningde, Fujian 352100, PR China
[b] Department of Mathematics, Ohio University, Zanesville Campus, Zanesville, OH 43701, USA

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ABSTRACT

Based on the notions of T. Banakh's strict Pytkeev networks and A.V. Arhangel'skiĭ's sensor families, strict Pytkeev networks with sensors are introduced in this paper. A family $\mathscr{P}$ of subsets of a topological space X is called a *strict Pytkeev network with sensors* (abbr. an *sp-network*) if, for each $x \in U \cap \overline{A}$ with U open and A subset in X, there is a set $P \in \mathscr{P}$ such that $x \in P \subset U$ and $x \in \overline{P \cap A}$. In present paper, we discuss certain relationship and operations among spaces defined by special Pytkeev networks, study spaces with a point-countable *sp*-network and spaces with a σ-closure-preserving *sp*-network, and detect some applications of *sp*-networks in topological groups.

The following results are obtained: (1) Every *sp*-network is preserved by a continuous pseudo-open mapping. (2) Every k-space with a point-countable *sp*-network coincides with a continuous pseudo-open s-image of a metric space. (3) Every regular feebly compact space with a point-countable *sp*-network has a point-countable base. (4) A regular space has a countable *sp*-network if and only if it is separable and has a point-countable *sp*-network. (5) A topological space is stratifiable if and only if it is a regular space with a σ-closure-preserving *sp*-network. (6) A regular space with a σ-locally finite *sp*-network has a σ-discrete *sp*-network. (7) A topological group is metrizable if it has countable *sp*-character. (8) There is a non-Fréchet-Urysohn sequential topological group with a countable strict Pytkeev network, which give a negative answer to a question posed by A.V. Arhangel'skiĭ [1].

1. Introduction

In 1983, E.G. Pytkeev [36] proved that every sequential space satisfies the following property, known actually as the Pytkeev property [33, Definition 0.2], which is stronger than countable tightness: a topological space X has the *Pytkeev property* if for each $A \subset X$ and each $x \in \overline{A} \setminus A$, there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ of

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* Corresponding author.
E-mail addresses: liuxintp@126.com (X. Liu), liuc1@ohio.edu (C. Liu), shoulin60@163.com (S. Lin).

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infinite subsets of A such that each neighborhood of x contains some A_n. B. Tsaban and L. Zdomskyy [38, Definition 5] strengthened this property as follows: a topological space X has the *strong Pytkeev property* if for each $x \in X$, there exists a countable family $\mathscr{P}$ of subsets of X, such that for each neighborhood U of x and each $A \subset X$ with $x \in \overline{A} \setminus A$, there is a set $P \in \mathscr{P}$ such that $P \subset U$ and $P \cap A$ is infinite. Clearly, the strong Pytkeev property implies countable cs^*-character [38, p. 8]. Meanwhile, in a study of Fréchet-Urysohn spaces, A.V. Arhangel'skiĭ [1] introduced and discussed sensitive families of a space, in which a countably sensitive family is exactly the strong Pytkeev property. These lead people to research the relations among the strong Pytkeev property, countability and generalized metrizable properties.

In 2011, A.V. Arhangel'skiĭ [2] further introduced sensor families of a topological space in a study of pseudo-open mappings. In 2015, the notions of Pytkeev networks and strict Pytkeev networks were introduced and studied [6], in which a Pytkeev network is just a network and a sensitive family in the sense of A.V. Arhangel'skiĭ [1]. T. Banakh [8, Proposition 1.8] proved that each countable (strict) Pytkeev network for a topological space is a k-network (resp. cs^*-network), and the converse is also true for a k-space (resp. Fréchet-Urysohn space). At the same time, the notions of *cp*-networks, *ck*-networks and *cn*-networks were introduced [18]. S.S. Gabriyelyan and J. Kąkol [18, Proposition 2.3] proved that each space with a countable *cn*-network at each point is of countable tightness. Various kinds of topological spaces have been defined by spaces with certain Pytkeev networks, which played an important role in generalized metric spaces, cardinal functions, function spaces, topological groups and topological vector spaces [2,6–8,17–20,38].

The following question was discussed in [18, p. 182].

Question 1.1. Under what circumstances can make some of the types of networks coincide?

Partial answers to this question were given by T. Banakh [6], S. Gabriyelyan and J. Kąkol [18], etc. They studied the relationship among countable families or σ-locally finite families with certain Pytkeev networks, k-networks and cs^*-networks. Recently, we further discussed the relationship among certain Pytkeev networks, strict Pytkeev networks, *cn*-networks and k-networks in a topological space, detect their operational properties, and raise some questions [32]. It is well known that for a regular space X, X has a σ-locally finite base (resp. network, k-network) if and only if it has a σ-discrete base [13, p. 282] (resp. network [23, Theorem 4.11], k-network [14, Theorem 4]). We have the following question.

Question 1.2. [32, Problem 4.9] Does every regular space with a σ-locally finite (strict) Pytkeev network have a σ-discrete (strict) Pytkeev network?

It is also known that every Fréchet-Urysohn regular space with a σ-closure-preserving k-network is stratifiable [31, Theorem 3.4.3], and every normal k-space with a σ-closure-preserving k-network is paracompact [31, Theorem 3.4.11]. We have the following question.

Question 1.3. [32, Problem 4.14] Is a normal space with a σ-locally finite strict Pytkeev network paracompact?

E.A. Michael [34, Theorem 11.4] proved that a regular space has a countable k-network (i.e., an $\aleph_0$-space) if and only if it is a compact-covering continuous image of a separable metrizable space.

Question 1.4. [18, Question 6.8] Find a characterization of regular spaces with a countable strict Pytkeev network analogous to the characterization of $\aleph_0$-spaces given by E.A. Michael.

On the other hand, A.V. Arhangel'skiĭ discussed some convergence concepts, components of first-countability and various kinds of pseudo-open mappings, and introduced the notions of a sensitive at a point family and a sensor at a set family [1,2]. A.V. Arhangel'skiĭ proved that every pseudocompact regular space has a point-countable base if it is a continuous pseudo-open s-image of a metric space [2, Theorem 2.13]. Does every pseudocompact regular space have a point-countable base if it is a quotient s-image of a metric space [29, Question 2.2.12]? Since a quotient s-image of a metric space coincides with a sequential space with a point-countable cs^*-network [31, Corollary 2.7.5], the following question is interesting.

Question 1.5. Does every pseudocompact regular sequential space with a point-countable cs^*-network have a point-countable base?

Spaces defined by special Pytkeev networks can be applied to topological spaces with certain algebra structures. T. Banakh and L. Zdomskyy proved that any sequential topological group G with countable cs^*-character is a stratifiable space with a σ-locally finite k-network, and any such G either is metrizable or contains an open submetrizable k_ω-subgroup [9, Theorem 1]. A.V. Arhangel'skiĭ pointed out that a Fréchet-Urysohn topological group need not be metrizable, and proved that if a topological group G is a Fréchet-Urysohn space and has countably sensitive at some point, then G is metrizable [1, Theorem 4.9]. Recently, T. Banakh and A. Leiderman proved that a locally narrow topological group has the strong Pytkeev property if and only if it is metrizable [8, Theorem 7]. The following question is posed.

Question 1.6. [1, p. 106] Is every countably-sensitive topological group metrizable?

In this paper, special Pytkeev networks are studied around the above questions. We will see that the concepts of strict Pytkeev networks and sensor families give us an inspiration for our research. In order to discuss generalized metrizable spaces with various base-like properties, and explore their applications in topological groups, we introduce and study a complex notion which is called a strict Pytkeev network with sensors (abbr. an sp-network), based on the notions of T. Banakh's strict Pytkeev networks and A.V. Arhangel'skiĭ's sensor families. Thus, by sp-networks, we obtain partial answers to Questions 1.1–1.5, and a negative answer to Question 1.6. The paper is organized as follows. In Section 2, we introduce some known notions to be discussed in this paper, and describe some basic relation among spaces defined by these notions. In Section 3, the notion of sp-networks is defined, and we obtain certain relationship among the notions of sp-networks, strict Pytkeev networks and k-networks, consider some standard operations in the spaces with certain sp-networks, give a special product property for spaces with countable sp-character, and prove that every sp-network is preserved by a continuous pseudo-open mapping. In Section 4, we study spaces with a point-countable sp-network, prove that a k-space with a point-countable sp-network if and only if it is a continuous pseudo-open s-image of a metrizable space, and each regular feebly compact space with a point-countable sp-network has a point-countable base, which give partial answers to Questions 1.4 and 1.5. In Section 5, we discuss spaces with a σ-closure-preserving sp-network, prove that a topological space is a stratifiable space if and only if it is a regular space with a σ-closure-preserving sp-network, and a regular space with a σ-locally finite sp-network has a σ-discrete sp-network, which give partial answers to Questions 1.2 and 1.3. In Section 6, we detect some applications of sp-networks in topological groups, prove that a topological group is metrizable if it has countable sp-character, and give a negative answer to Question 1.6.

2. Networks and countable tightness

In this section, we introduce the necessary notation, terminology, and describe some basic relation among spaces defined by these notions. Throughout this paper, all topological spaces are assumed to be Hausdorff, mappings are continuous and onto, unless explicitly stated otherwise. The sets of real numbers, rational numbers and positive integers are denoted by $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$, respectively. The first infinite ordinal and the first uncountable ordinal are denoted by ω and ω_1, respectively. For a set A, denote $A^{<\omega} = \{B \subset A : B \text{ is finite}\}$.

Definition 2.1. Let $\mathscr{P}$ be a family of subsets of a topological space X.

(1) $\mathscr{P}$ is a *network* [13, p. 127] for X, if for any neighborhood U of a point $x \in X$, there exists a set $P \in \mathscr{P}$ such that $x \in P \subset U$.

(2) $\mathscr{P}$ is a k*-network* [23, Definition 11.1] for X, if whenever K is a compact subset of an open set U in X, there exists a finite subfamily $\mathscr{F}$ of $\mathscr{P}$ such that $K \subset \bigcup \mathscr{F} \subset U$.

(3) $\mathscr{P}$ is a cs^**-network* [21, Definition 3] (resp. wcs^**-network* [30, p. 79]) for X, if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to a point $x \in U$ with U open in X, there exists a set $P \in \mathscr{P}$ such that some subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is eventually in P and $x \in P \subset U$ (resp. $P \subset U$).

(4) $\mathscr{P}$ is a *Pytkeev network* (resp. *strict Pytkeev network*) [6, Definition 1.1] for X, if $\mathscr{P}$ is a network for X, and for each neighborhood U of a point x in X and each subset A of X accumulating at x, there exists a set $P \in \mathscr{P}$ such that $P \cap A$ is infinite and $P \subset U$ (resp. $x \in P \subset U$).

For a fixed point $x \in X$, $\mathscr{P}$ is called a *network* (resp. cs^**-network* (wcs^**-network*), *Pytkeev network* (*strict Pytkeev network*)) of the point x in X, if the family $\mathscr{P}$ satisfies the above mentioned conditions (1) (resp. (3) or (4)) at x.

(5) The space X is of *countable* cs^**-character* [9, p. 26] (resp. the *strong Pytkeev property* [38, Definition 5] or *countable Pytkeev character*) if it has a countable cs^*-network (resp. Pytkeev network) at each point $x \in X$.

Remark 2.2. (1) In [6, p. 152], it was said that a subset A of a topological space X *accumulates at a point* $x \in X$ if each neighborhood of x contains infinitely many points of the set A. It is obvious that for a T_1 space X a point $x \in X$ is an accumulation point of a set $A \subset X$ if and only if $x \in \overline{A \setminus \{x\}}$.

(2) In [1, p. 104], the notion of sensitive families was introduced. Let X be a T_1 topological space, and x be a point of X. A family $\mathscr{S}$ of subsets of X is called *sensitive* at x (or just *closure-sensitive* at x) if, for each neighborhood U of x and for each subset A of $X \setminus \{x\}$ such that $x \in \overline{A}$, there exists a set $P \in \mathscr{S}$ satisfying the following conditions: $P \subset U$ and $P \cap A$ is infinite. It is clear that $\mathscr{S}$ is a Pytkeev network of x if and only if it is a network and a sensitive family at x. Thus, a space X has the strong Pytkeev property if and only if X has a network which is countably sensitive at each point of X.

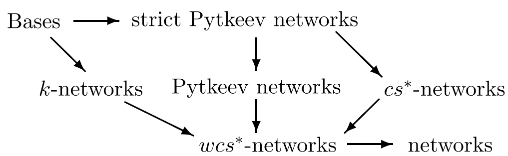

Fig. 1. The relationship among spaces with certain networks.

Fig. 1 is some basic relationship among spaces with certain networks to be discussed in this paper [6,18].

Spaces defined by a countable special network are a matter of continuing concern. Let X be a regular space. X is called a *cosmic space* [34, p. 993] if it has a countable network; X is called an $\aleph_0$*-space* [34, Definition 1.2] if it has a countable k-network; and X is called a $\mathfrak{P}_0$*-space* [6, Definition 1.2] if it has a countable Pytkeev network. It is easy to check that every $\mathfrak{P}_0$-space is an $\aleph_0$-space, and every $\aleph_0$-space is a cosmic space. Some properties of the spaces with a countable special network are the source of our further research on the spaces with a point-countable special network or a σ-locally finite special network. The further relationship among spaces with certain networks involves weak first-countability.

Definition 2.3. Let X be a topological space.

(1) A subset A of X is k*-closed* in X if $A \cap K$ is relatively closed in K for each compact subset K of X. The space X is a k*-space* [13, p. 152] if every k-closed subset of X is closed.

(2) A subset A of X is *sequentially closed* in X if S is a sequence in A converging to a point $x \in X$, then $x \in A$. The space X is a *sequential space* [13, p. 53] if every sequentially closed subset of X is closed.

(3) The space X is a *Fréchet-Urysohn space* [13, p. 53] if, for each $A \subset X$ and each $x \in \overline{A}$, there is a sequence in A converging to the point x in X.

(4) The space X is of *countable tightness* [35, Proposition 8.5] if, for each $A \subset X$ and each $x \in \overline{A}$, there is a countable subset $C \subset A$ such that $x \in \overline{C}$.

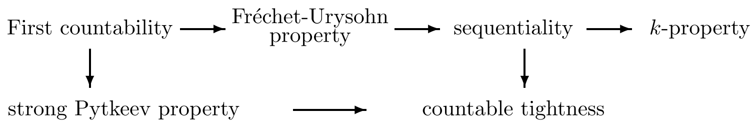

Fig. 2. The relationship among certain countability.

Fig. 2 is some basic relationship among certain countability to be discussed in this paper [18].

The following are known.

Lemma 2.4. (1) *Every k-network in a k-space is a Pytkeev network [6, Proposition 1.7].*

(2) *Every wcs^*-network in a sequential space is a Pytkeev network [32, Theorem 3.5].*

The above lemma shows that weak first-countability plays an important role in the study of spaces with special Pytkeev networks. On the other hand, there are some interesting results from the spaces with a point-countable special Pytkeev network. A family $\mathscr{P}$ of subsets of a space X is *point-countable* [11, p. 350] if the family $\{P \in \mathscr{P} : x \in P\}$ is countable for each $x \in X$. A space X is called a *meta-Lindelöf space* [11, p. 370] if every open cover of X has a point-countable open refinement.

Lemma 2.5. (1) *Every k-space with a point-countable k-network is a sequential space [25, Corollary 3.4].*

(2) *Every first-countable space with a point-countable cs^*-network has a point-countable base [31, Corollary 2.7.18].*

(3) *Every point-countable Pytkeev network for a space is a k-network [32, Theorem 3.1].*

(4) *Every space with a point-countable strict Pytkeev network is a hereditarily meta-Lindelöf space [32, Corollary 4.3].*

3. Strict Pytkeev networks with sensors

In this section, we introduce and study a complex notion which is called a strict Pytkeev network with sensors (abbr. an *sp*-network), based on the notions of T. Banakh's strict Pytkeev networks and A.V. Arhangel'skiĭ's sensor families.

A family $\mathscr{S}$ of subsets of a topological space X is said to be a *sensor* [2, p. 217] at a set $H \subset X$ if, for each open neighborhood $O(H)$ of H and each set A in X such that $H \cap \overline{A} \neq \varnothing$, there exists a set $P \in \mathscr{S}$ satisfying the following conditions: $P \subset O(H)$ and $H \cap \overline{A \cap P} \neq \varnothing$. Thus, for a point $x \in X$, $\mathscr{S}$ is a sensor at x if, for each neighborhood U of x and each set A with $x \in \overline{A}$ in X, there exists a set $P \in \mathscr{S}$ such that $P \subset U$ and $x \in \overline{A \cap P}$. This motivates us to propose the following concept.

Definition 3.1. A family $\mathscr{P}$ of subsets of a topological space X is called a *strict Pytkeev network with sensors* (abbr. an *sp-network*) for X if, for each $x \in U \cap \overline{A}$ with U open and A subset in X, there is a set $P \in \mathscr{P}$ such that $x \in P \subset U$ and $x \in \overline{P \cap A}$.

For a fixed point $x \in X$, $\mathscr{P}$ is called an *sp-network* of the point x in X if the family $\mathscr{P}$ satisfies the above mentioned conditions at x. The space X is of *countable sp-character at x* if it has a countable *sp*-network at the point $x \in X$.

A topological space X is said to be a *stric $\mathfrak{P}_0$-space* if X is regular and has a countable *sp*-network.

Remark 3.2. Maybe we can call the term of Definition 3.1 a "strict sensor", but we would prefer to call it a "strict Pytkeev network with sensors". The term "strict Pytkeev network with sensors" is too long, and we think about giving it an abbreviation. It is natural that the phrase "strict Pytkeev network" is abbreviated to "*sp*-network", and it may be considered that the phrase "strict Pytkeev network with sensors" is abbreviated to "*sps*-network" or "*ssp*-network". For simplicity, we use the word "*sp*-network" to represent the concept of "strict Pytkeev network with sensors" in this paper. The notion of *cp*-networks was introduced in [18, Definition 1.1]. A family $\mathscr{P}$ of subsets of a space X is called a *cp-network* at a point $x \in X$ if either x is an isolated point of X and $\{x\} \in \mathscr{P}$, or for each subset A of X with $x \in \overline{A} \setminus A$ and each neighborhood U of x there exists a set $P \in \mathscr{P}$ such that $P \cap A$ is infinite and $x \in P \subset U$. It is easy to check that for a T_1 space X a family $\mathscr{P}$ of subsets of X is a strict Pytkeev network if and only if it is a *cp*-network for X. Thus, the concept of a strict Pytkeev network has a similar abbreviation.

It is clear that every base for a topological space is an *sp*-network; every *sp*-network for a topological space is a strict Pytkeev network (i.e., a *cp*-network), and the converse is also true for a Fréchet-Urysohn space, which extends the following result: every cs^*-network is a strict Pytkeev network in a Fréchet-Urysohn space [8, Proposition 1.8].

Lemma 3.3. *Every cs^*-network in a Fréchet-Urysohn space is an sp-network.*

Proof. Let X be a Fréchet-Urysohn space, and $\mathscr{P}$ be a cs^*-network for X. Given a point $x \in O \cap \overline{A}$ with O open and A subset in the space X, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A converging to the point x in X, thus there is a set $P \in \mathscr{P}$ such that some subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is eventually in P and $x \in P \subset U$. Then $x \in \overline{P \cap A}$. Hence, the family $\mathscr{P}$ is an *sp*-network for X. □

A topological space X has *countable closed pseudocharacter* [13, p. 135] or *regular* G_δ [28, p. 188] at a point $x \in X$ if $\{x\} = \bigcap_{n\in\omega} U_n$ for some sequence $\{U_n\}_{n\in\omega}$ of closed neighborhoods of x. A space X is *discretely sequential* at a point $x \in X$ if for any discrete subspace $D \subset X \setminus \{x\}$ with $\overline{D} = \{x\} \cup D$ there is a sequence of points $\{x_n\}_{n\in\omega} \subset D$ that converges to x.

Theorem 3.4. *Assume that a T_1-space X has countable closed pseudocharacter and countable sp-networks at a point $x \in X$. The space X is Fréchet-Uryshon at x if and only if X is discretely sequential at x.*

Proof. The "only if" part is trivial. To prove the "if" part, assume that the space X is discretely sequential at x. Let $\mathscr{P}$ be a countable *sp*-network at x and $\{W_n\}_{n\in\omega}$ be a decreasing sequence of closed neighborhoods of x such that $\{x\} = \bigcap_{n\in\omega} W_n$. To prove that X is Fréchet-Uryshon at x, take any subset $A \subset X$ with $x \in \overline{A} \setminus A$. Consider the subfamily $\mathscr{P}' = \{P \in \mathscr{P} : x \in \overline{P \cap A}\}$ and write it as $\mathscr{P}' = \{P_k\}_{k\in\omega}$. For every $k \in \omega$ choose a point $a_k \in W_k \cap P_k \cap A$. If the set $D = \{a_k\}_{k\in\omega}$ is closed in X, then $X \setminus D$ is a neighborhood of x and we can find a set $P \in \mathscr{P}$ such that $x \in \overline{P \cap A}$ and $P \subset X \setminus D$. It follows that $P \in \mathscr{P}'$ and hence $P = P_k$ for some $k \in \omega$. Then $a_k \in P_k \cap D \subset (X \setminus D) \cap D = \varnothing$, which is a contradiction showing that the set D is not closed in X.

We claim that the point x is a unique accumulation point of D. Indeed, assuming that D has an accumulation point $y \neq x$, we can find $n \in \omega$ such that $y \notin W_n$ and conclude that $X \setminus W_n$ is an open neighborhood of y such that $D \cap (X \setminus W_n) \subset \{a_i\}_{i<n}$ which means that y is not an accumulation point of D. This contradiction shows that D is a discrete subspace of $X \setminus \{x\}$ and $\overline{D} = \{x\} \cup D$. By the discrete sequentiality of X, there exists a sequence $\{x_n\}_{n\in\omega} \subset D \subset A$ that converges to x. □

Corollary 3.5. *If a space X has countable closed pseudocharacter and countable sp-networks at each point, then each k-subspace of X is Fréchet-Uryshon.*

Proof. Let Y be a k-subspace of the space X and a point $y \in Y$. Since each point of Y has countable closed pseudocharacter, the subspace Y is sequential. If D is a discrete subspace of $Y \setminus \{y\}$ with $\mathrm{cl}_Y D = \{y\} \cup D$, then D is not sequentially closed in Y, thus there is a sequence of points $\{x_n\}_{n\in\omega} \subset D$ that converges to y. This shows that each point in Y is discretely sequential. By Theorem 3.4, Y is Fréchet-Uryshon. □

Not every regular space with a countable *sp*-network is a k-space.

Example 3.6. There exists a countable regular space X with a unique non-isolated point such that X has a countable *sp*-network but contains no infinite compact subsets.

Proof. Let $2^{\leq\omega} = 2^\omega \cup 2^{<\omega}$ where $2^{<\omega} = \bigcup_{n\in\omega} 2^n$ is the family of finite binary sequences.

For a sequence $s \in 2^n$ by $|s| = n$ we denote its length. Let also $\uparrow s := \{t \in 2^{\leq\omega} : |t| \geq |s| \text{ and } t \upharpoonright |s| = s\}$. Endow $2^{\leq\omega}$ with the compact second-countable topology generated by the base $\{\uparrow s : s \in 2^{<\omega}\}$. Let $\mathscr{U}$ be the family of open subsets $U \subset 2^{\leq\omega}$ such that $2^\omega \setminus U$ is finite. The family $\mathscr{U}$ induces the filter $\mathscr{F} = \{U \cap 2^{<\omega} : U \in \mathscr{U}\}$ on the countable set $2^{<\omega}$.

Take any point $\infty \notin 2^{<\omega}$ and consider the space $X = \{\infty\} \cup 2^{<\omega}$ endowed with the topology as follows:

$$\tau = \{U \subset X : \infty \in U \Rightarrow U \cap 2^{<\omega} \in \mathscr{F}\}.$$

It can be shown that X has the required properties: it contains no infinite compact sets and the family

$$\mathscr{P} = \{\{s\} : s \in 2^{<\omega}\} \cup \{\{\infty\} \cup \uparrow s \setminus 2^\omega : s \in 2^{<\omega}\}$$

is a countable *sp*-network in X. □

The condition "cs^*-network" in Lemma 3.3 cannot be replaced by "Pytkeev network", see Example 4.6. The following example shows that the condition "Fréchet-Urysohn space" in Lemma 3.3 cannot be weakened to "sequential space".

Example 3.7. The Arens space S_2: a regular sequential space with a countable strict Pytkeev network which is not an *sp*-network.

Proof. Let $X = \{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{n,m} : n, m \in \mathbb{N}\}$, where every x_n, $x_{n,m}$ and x are different from each other. The set X endowed with the following topology is called the *Arens space* [13, Example 1.6.19] and denoted briefly as S_2: each $x_{n,m}$ is isolated; a basic neighborhood of x_n has the form $\{x_n\} \cup \{x_{n,m} : m > k\}$ for some $k \in \mathbb{N}$; a basic neighborhood of x has the form $\{x\} \cup \bigcup_{n\geq k} V_n$ for some $k \in \mathbb{N}$, where each V_n is a neighborhood of x_n. It is easy to see that the space X is a non-Fréchet-Urysohn, regular sequential space with a countable cs^*-network [31, Example 1.8.6]. By Lemma 2.4(2), X has a countable Pytkeev network, and it has a countable strict Pytkeev network [6, p. 152].

Since every countable space has countable closed pseudocharacter at each point, by Corollary 3.5, the space X does not have countable character at the point x. Next, we will directly show that the space X does not have a countable *sp*-network at the point x. Otherwise, we can assume that $\mathscr{F}$ is a countable *sp*-network at x. Let $X_0 = \{x_{n,m} : n, m \in \mathbb{N}\}$. Put $\mathscr{F}' = \{F \in \mathscr{F} : x \in \overline{F \cap X_0}\}$. If $F \in \mathscr{F}'$, the set $\{n \in \mathbb{N} : x_{n,m} \in F \text{ for some } m \in \mathbb{N}\}$ is infinite. Thus there exists a subset C of X_0 such that $|C \cap \{x_{n,m} : m \in \mathbb{N}\}| \leq 1$ for each $n \in \mathbb{N}$ and $|C \cap F| = 1$ for each $F \in \mathscr{F}'$. Since $x \in (X \setminus C) \cap \overline{X_0}$ and C is closed in X, there is a set $F \in \mathscr{F}$ such that $x \in F \subset X \setminus C$ and $x \in \overline{F \cap X_0}$, which is a contradiction. Therefore, X does not have a countable *sp*-network at the point x, and X does not have the strong Pytkeev property. □

Example 3.8. There exists a compact space possessing an *sp*-network which is not a k-network.

Proof. Let $X = [0, \omega_1]$ endowed with the ordered topology. Then X is a compact space. Let L be the set of all limit ordinals in X. For each $\alpha < \omega_1$ and each $n < \omega$, let $P_{\alpha,n} = (\{\beta + n : \beta \in L\} \cap (\alpha, \omega_1]) \cup \{\omega_1\}$. Clearly, $(\alpha, \omega_1] = \bigcup_{n<\omega} P_{\alpha,n}$. For each $x \in X$, define a family $\mathscr{P}_x$ of subsets of X as follows: if $x \neq \omega_1$, let $\mathscr{P}_x$ be a countable local base of x with $\bigcup \mathscr{P}_x \subset [0, x]$; if $x = \omega_1$, let $\mathscr{P}_x = \{P_{\alpha,n} : \alpha < \omega_1, n < \omega\}$. Put $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$. Since X is a compact space which is not covered by any finitely many elements of $\mathscr{P}$, the family $\mathscr{P}$ is not a k-network for X. We will prove that $\mathscr{P}$ is an *sp*-network for X. Given a point $x \in U \cap \overline{A}$ with U open and A subset in X, without loss of generality, we may assume that $x = \omega_1 \in \overline{A} \setminus A$, there is an ordinal $\alpha < \omega_1$ such that $(\alpha, \omega_1] \subset U$. Then $A \cap (\alpha, \omega_1]$ is an uncountable set because x is an accumulation point of the set A. By $A \cap (\alpha, \omega_1] = \bigcup_{n<\omega} A \cap P_{\alpha,n}$, there exists $n < \omega$ such that $A \cap P_{\alpha,n}$ is uncountable. Then $x \in \overline{A \cap P_{\alpha,n}}$, and $x \in P_{\alpha,n} \subset U$. Therefore, $\mathscr{P}$ is an *sp*-network for X. □

Next, we consider some standard operations in the topological spaces with certain *sp*-networks. It is obvious that (1) if $\mathscr{P}$ is an *sp*-network for a space X and Y is a subspace of X, then $\mathscr{P}|_Y = \{P \cap Y : P \in \mathscr{P}\}$ is an *sp*-network for Y; (2) if $\{X_\alpha\}_{\alpha\in\Gamma}$ is a family of spaces, and $\mathscr{P}_\alpha$ is an *sp*-network for X_α for each $\alpha \in \Gamma$, then the family $\bigcup_{\alpha\in\Gamma} \mathscr{P}_\alpha$ is an *sp*-network for the topological sum $\bigoplus_{\alpha\in\Gamma} X_\alpha$.

Theorem 3.9. *Let x, y be non-isolated points of spaces X, Y, respectively. If the product $X \times Y$ has a countable sp-network at the point $(x, y) \in X \times Y$, then $X \times Y$ is first-countable at (x, y).*

Proof. Assume that the space $X \times Y$ has a countable *sp*-network at (x, y). The first-countability of $X \times Y$ at (x, y) will be proved as soon as we proved the first-countability of X and Y at x and y, respectively.

By symmetry, it suffices to present a proof of the first-countability of X at x. Taking into account that the spaces X, Y can be identified with the subspaces $\{x\} \times Y$ and $X \times \{y\}$ of $X \times Y$, we conclude that X and Y have countable *sp*-networks at x and y, respectively. In particular, the space Y is countably tight at y, which allows to find a sequence of points $\{y_n\}_{n\in\omega} \subset Y \setminus \{y\}$ accumulating at y. Since the space Y is Hausdorff, for every $n \in \omega$ we can choose a closed neighborhood $W_n \subset Y$ of y_n such that $y \notin W_n$.

Let $\mathscr{P}$ be a countable *sp*-network at the point (x, y) of the space $X \times Y$. Replacing $\mathscr{P}$ by a larger family, we can assume that the family $\mathscr{P}$ is closed under finite unions. Let $\mathrm{pr}_X : X \times Y \to X$ denote the natural projection onto the first factor. The first-countability of X at x will be established as soon as we checked that for every neighborhood $O_x \subset X$ of x there exists a set $P \in \mathscr{P}$ such that $\mathrm{pr}_X(P) \subset O_x$ and $\mathrm{pr}_X(P)$ is a neighborhood of x.

Given any neighborhood $O_x \subset X$ of x, consider the subfamily $\mathscr{P}' = \{P \in \mathscr{P} : P \subset O_x \times Y\}$, which can be written as $\mathscr{P}' = \{P_k\}_{k\in\omega}$. We claim that for some $k \in \omega$ the projection $\mathrm{pr}_X(P_k)$ is a neighborhood of x.

To derive a contradiction, assume that for every $k \in \omega$ the set $\mathrm{pr}_X(P_k)$ is not a neighborhood of x. Since the family $\mathscr{P}'$ is closed under finite unions, for every $n \in \omega$ the set $\mathrm{pr}_X(\bigcup_{k\leq n} P_k)$ is not a neighborhood of x and hence $\bigcup_{k\leq n} P_k$ is not a neighborhood of the point (x, y_n), which implies that the point (x, y_n) is contained in the closure of the set $(X \times W_n) \setminus \bigcup_{k\leq n} P_k$.

Since the point (x, y) is an accumulation point of the sequence $\{(x, y_n)\}_{n\in\omega}$, the set

$$A = \bigcup_{n\in\omega} ((X \times W_n) \setminus \bigcup_{k\leq n} P_k)$$

contains (x, y) in its closure. Since $\mathscr{P}$ is an *sp*-network at (x, y), there exists a set $P \in \mathscr{P}$ such that $P \subset O_x \times Y$ and $(x, y) \in \overline{P \cap A}$. It follows that $P \in \mathscr{P}'$ and hence $P = P_i$ for some $i \in \omega$.

Then

$$P \cap A = \bigcup_{n\in\omega} (P_i \cap (X \times W_n) \setminus \bigcup_{k\leq n} P_k) \subset \bigcup_{n<i} X \times W_n$$

and hence (x, y) cannot belong to the closure of $P \cap A$ as it does not belong to the closure of the set $\bigcup_{n<i} X \times W_n$. □

Recall some concepts relevant to mappings. A mapping $f : X \to Y$ between topological spaces is called a *quotient mapping* [13, p. 91] if, for each $U \subset Y$ with $f^{-1}(U)$ open in X, U is open in Y. A mapping $f : X \to Y$ is *pseudo-open* [1, p. 108] if for each $f^{-1}(y) \subset V$ with V open in X, $f(V)$ is a neighborhood of y in Y. Obviously, every closed mapping or open mapping is a pseudo-open mapping, and every pseudo-open mapping is a quotient mapping.

Theorem 3.10. *Every sp-network is preserved by a pseudo-open mapping.*

Proof. Let $\mathscr{P}$ be an *sp*-network for a space X and $f : X \to Y$ be a pseudo-open mapping. Given a point $y \in O \cap \overline{A}$ with O open and A subset in the space Y, then $f^{-1}(y) \cap \overline{f^{-1}(A)} \neq \varnothing$. Otherwise, we have $f^{-1}(y) \subset X \setminus \overline{f^{-1}(A)}$. Because f is a pseudo-open mapping, $y \in [f(X \setminus \overline{f^{-1}(A)})]^{\circ} \subset Y \setminus \overline{A}$, which is a contradiction. So there exist a point $x \in f^{-1}(y) \cap \overline{f^{-1}(A)}$ and a neighborhood V of x such that $f(V) \subset O$. Since $\mathscr{P}$ is an *sp*-network for X, there exists a set $P \in \mathscr{P}$ such that $x \in P \subset V$ and $x \in \overline{P \cap f^{-1}(A)}$. So $y = f(x) \in f(\overline{P \cap f^{-1}(A)}) \subset \overline{f(P) \cap A}$ and $y \in f(P) \subset O$. Hence, the family $\mathscr{Q} = \{f(P) : P \in \mathscr{P}\}$ is an *sp*-network for Y. □

sp-Networks may not be preserved by quotient mappings, see Example 4.7. The following corollary is obvious.

Corollary 3.11. *Spaces with a countable sp-network are preserved by pseudo-open mappings.*

4. Spaces with point-countable *sp*-networks

In this section, we mainly discuss some properties of spaces with a point-countable *sp*-network, and prove that every regular feebly compact space with a point-countable *sp*-network has a point-countable base.

Let X, Y be topological spaces. A mapping $f : X \to Y$ is called an *s-mapping* [25, p. 304] if every $f^{-1}(y)$ is a separable subset of X. It is well known that some spaces with point-countable special networks can be characterized by certain *s*-images of metric spaces [31]. For example, a space with a point-countable base if and only if it is an open *s*-image of a metric space [31, Theorem 2.7.17]; a sequential space with a point-countable cs^*-network if and only if it is a quotient *s*-image of a metric space [31, Corollary 2.7.5]. The following theorem establishes a relation between spaces with point-countable *sp*-networks and metric spaces by pseudo-open *s*-mappings, which gives a partial answer to Question 1.4.

Lemma 4.1. *[28, Corollary 2.13] Suppose that X is a k-space with a point-countable k-network. Then X is a Fréchet-Urysohn space if and only if X contains no closed copy of S_2.*

Theorem 4.2. *The following conditions are equivalent for a topological space X.*
(1) *X is a k-space with a point-countable sp-network.*
(2) *X is a Fréchet-Urysohn space with a point-countable cs^*-network.*
(3) *X is a pseudo-open s-image of a metrizable space.*

Proof. It is known that (2) is equivalent to (3) [31, Corollary 2.7.5]. (2) implies (1) by Lemma 3.3. Next, we prove that (1) implies (2). Let X be a k-space with a point-countable *sp*-network. Obviously, X has a point-countable cs^*-network. By Example 3.7, X contains no closed copy of S_2. It follows from Lemmas 2.5(3) and 4.1 that X is a Fréchet-Urysohn space. □

Let $\mathscr{P}$ be a cover of a topological space X. The space X is *determined by* the cover $\mathscr{P}$ [25, p. 303], if $U \subset X$ is open (closed) in X if and only if $U \cap P$ is relatively open (relatively closed) in P for each $P \in \mathscr{P}$. It is known that if $\mathscr{P}$ is a cover of X, then the space X is determined by $\mathscr{P}$ if and only if the obvious mapping $f : \bigoplus \mathscr{P} \to X$ is a quotient mapping, where $\bigoplus$ denotes the topological sum [25, Lemma 1.8]. A topological space X is called a *k_ω-space* [16, p. 111] if it is determined by a countable cover of compact subsets of X. It is clear that every k_ω-space is a k-space.

The following example shows that spaces with a countable *sp*-network are not preserved by finite Tychonoff products.

Example 4.3. The sequential fan S_ω has the following properties:

(1) S_ω is a regular Fréchet-Urysohn, k_ω-and $\aleph_0$-space.
(2) S_ω has a countable *sp*-network.
(3) $(S_\omega)^2$ does not have a point-countable *sp*-network.

Proof. A topological space X is called the *sequential fan* [3, p. 316], which is denoted briefly as S_ω, if X is the quotient space by identifying all the limit points of ω many non-trivial convergent sequences. Since the space S_ω is a closed image of a separable locally compact metrizable space, it is easy to check that S_ω is a regular Fréchet-Urysohn, k_ω-and $\aleph_0$-space [31, Example 1.8.7]. By Corollary 3.11, S_ω has a countable *sp*-network.

Since S_ω is not first-countable, by Theorem 3.9, $(S_\omega)^2$ does not have a point-countable *sp*-network. □

By Theorem 3.9 and Lemma 2.5(2), the following is obtained.

Corollary 4.4. *A topological space X has a point-countable base if and only if X^2 has a point-countable sp-network.*

A mapping $f : X \to Y$ is called *countable-to-one* [25, p. 317] (resp. *finite-to-one* [11, p. 388]) if every $f^{-1}(y)$ is a countable (resp. finite) subset of X.

Theorem 4.5. *Every space with a point-countable sp-network is preserved by a countable-to-one pseudo-open mapping.*

Proof. Let $f : X \to Y$ be a countable-to-one pseudo-open mapping, and $\mathscr{P}$ be a point-countable *sp*-network for the space X. By Theorem 3.10, $\{f(P) : P \in \mathscr{P}\}$ is an *sp*-network for the space Y. Because f is a countable-to-one mapping, $\{f(P) : P \in \mathscr{P}\}$ is a point-countable *sp*-network for Y. □

Example 4.6. Spaces with a point-countable *sp*-network are not preserved by closed mappings.

Proof. Let S_{ω_1} be the quotient space obtained by identifying all the limit points of the topological sum of ω_1 many non-trivial convergent sequences. Obviously, the space S_{ω_1} is a closed image of a metrizable space. Thus, S_{ω_1} is a Fréchet-Urysohn space with a point-countable k-network [31, Theorem 2.5.8]. By Lemma 2.4(1), S_{ω_1} has a point-countable Pytkeev network. It is clear that S_{ω_1} is a closed image of a space with a point-countable *sp*-network. It follows from [31, Example 1.8.7] that S_{ω_1} does not have a point-countable cs^*-network, thus it does not have a point-countable *sp*-network (see Fig. 1). □

Example 4.7. Spaces with a point-countable *sp*-network are not preserved by finite-to-one quotient mappings.

Proof. Let

$$\mathbb{I} = [0,1], S_1 = \{1/n : n \in \mathbb{N}\} \cup \{0\}, X = \mathbb{I} \times S_1, \text{ and } Y = \mathbb{I} \times (S_1 \setminus \{0\}).$$

Define a topology for X as follows [25, Example 9.3]: Y is the Euclidean subspace of X; a basic neighborhood of a point $(t, 0) \in X$ has the form

$$\{(t,0)\} \cup \bigcup\{V(t,k) : k \geq n\}, \quad n \in \mathbb{N},$$

where each $V(t,k)$ is an open neighborhood of the point $(t, 1/k)$ in the subspace $\mathbb{I} \times \{1/k\}$.

Let

$$M = (\bigoplus\{\mathbb{I} \times \{1/n\} : n \in \mathbb{N}\}) \oplus (\bigoplus\{\{t\} \times S_1 : t \in \mathbb{I}\}).$$

Then M is a locally compact metrizable space, thus M has a point-countable *sp*-network. Let $f : M \to X$ be the obvious mapping. Since X is determined by the point-finite cover $\{\mathbb{I}\times\{1/n\} : n \in \mathbb{N}\} \bigcup\{\{t\} \times S_1 : t \in \mathbb{I}\}$, f is a finite-to-one quotient mapping [25, Lemma 1.8].

Obviously, the space X is a separable regular space. Since $\mathbb{I} \times \{0\}$ is an uncountable discrete closed subspace of X, X is not a Lindelöf space, and hence X is not a meta-Lindelöf space. By Lemma 2.5(4), X does not have a point-countable strict Pytkeev network. Hence, X does not have a point-countable *sp*-network. □

Let X be a topological space. A family $\mathscr{P}$ of subset of X is called *locally finite* [11, p. 349] if for each point x in X there is a neighborhood O_x of the point x such that the set O_x meets at most finitely many elements of the family $\mathscr{P}$. A topological space X is called *feebly compact* [37, p. 482] if every locally finite family of open sets of X is finite; X is called *pseudocompact* if every real-valued continuous function on X is bounded. It is well known that every countably compact space is feebly compact, every feebly compact space is pseudocompact, and every completely regular pseudocompact space is feebly compact. By [32, Corollary 3.4], each countably compact space with a point-countable Pytkeev network has a point-countable base. The following result gives a partial answer to Question 1.5.

Lemma 4.8. *Let X be a regular feebly compact space and $x \in X$. If X has a countable sp-network at the point x, then it is first-countable at x.*

Proof. Let $\mathscr{P}$ be a countable *sp*-network at the point x. Put

$$\mathscr{B} = \{\overline{\bigcup \mathscr{P}'} : \mathscr{P}' \in [\mathscr{P}]^{<\omega}\}.$$

Then $\mathscr{B}$ is countable. Next we will show that $\{B \in \mathscr{B} : x \in B^\circ\}$ is a local base of the point x.

Take any open neighborhood V of x. Let $\mathscr{Q} = \{\overline{P} \subset V : P \in \mathscr{P}\}$, which can be written as $\mathscr{Q} = \{Q_n\}_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, let $B_n = \bigcup\{Q_i : i \leq n\}$. Then $x \in Q_n \subset B_n \subset V$ and $B_n \in \mathscr{B}$. Put $A_n = X \setminus B_n$. Then A_n is open in X. Let

$$S = \{s \in X : \text{there exists a sequence } \mathscr{W}_s = \{W_i\}_{i\in\mathbb{N}} \text{ of open subsets of } X \text{ such that } \mathscr{W}_s \text{ is not locally finite at the point } s, W_i \subset A_i \text{ and } s \notin \overline{W_i} \text{ for each } i \in \mathbb{N}\}.$$

Claim 1. *If $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, then $x \in \overline{S}$.*

Let O be an open neighborhood of the point x. There exists an open neighborhood G of x such that $\overline{G} \subset O$. Since X has a countable *sp*-network at x, X has countable tightness at x [18, Proposition 2.3]. For each $n \in \mathbb{N}$, by $x \in \overline{A_n}$, there is a countable subset $D_n \subset A_n$ such that $x \in \overline{D_n}$. Since $x \notin A_n$, D_n is infinite. Denote $D_n = \{d_{n,i} : i \in \mathbb{N}\}$. Since A_n is open, we can select a sequence $\{W_{n,i}\}_{i\in\mathbb{N}}$ of open subsets of X such that $x \notin \overline{W_{n,i}}$ and $d_{n,i} \in W_{n,i} \subset A_n$. Then $x \in \overline{\bigcup_{i\in\mathbb{N}} W_{n,i}}$, and we can choose $i(n) \in \mathbb{N}$ such that

$G \cap W_{n,i(n)} \neq \varnothing$. Because X is a feebly compact space, there exists $s \in X$ such that $\{G \cap W_{n,i(n)}\}_{n\in\mathbb{N}}$ is not locally finite at the point s. Thus $s \in S \cap \overline{G} \subset S \cap O$. Therefore, $x \in \overline{S}$.

Claim 2. *$x \in B_n^\circ$ for some $n \in \mathbb{N}$.*

Suppose that the claim is not hold, $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. By Claim 1, $x \in \overline{S}$. Since X is of countable tightness at the point x, there exists a subset $\{s_n : n \in \mathbb{N}\} \subset S$ such that $x \in \overline{\{s_n : n \in \mathbb{N}\}}$. For each $n \in \mathbb{N}$, there exists a sequence $\mathscr{H}_n = \{H_{n,i}\}_{i\in\mathbb{N}}$ of open subsets of X such that $\mathscr{H}_n$ is not locally finite at the point s_n, each $H_{n,i} \subset A_i$ and $x \notin \overline{H_{n,i}}$. Let $H = \bigcup\{H_{n,i} : n, i \in \mathbb{N} \text{ and } n \leq i\}$. If U is an open neighborhood of the point x in X, there exists $s_n \in U$. Since the family $\mathscr{H}_n$ is not locally finite at the point s_n, there is $k \geq n$ such that $U \cap H_{n,k} \neq \varnothing$, and $U \cap H \neq \varnothing$. This shows that $x \in \overline{H}$. Because $\mathscr{P}$ is an *sp*-network at x, there exists $i \in \mathbb{N}$ such that $x \in \overline{Q_i \cap H}$. Since $B_i \cap H_{n,j} \subset B_i \cap A_i = \varnothing$ for each $j \geq i$, $x \in \overline{B_i \cap H} \subset \bigcup\{\overline{H_{n,j}} : n \leq j < i\}$, which is a contradiction.

It shows that $\{B \in \mathscr{B} : x \in B^\circ\}$ is a countable local base of the point x. □

Corollary 4.9. *Let X be a regular feebly compact space. If X has a point-countable sp-network, then it has a point-countable base.*

Proof. Since X has a point-countable *sp*-network, by Theorem 4.8, X is a first-countable space. Hence, by Lemma 2.5(2), X has a point-countable base. □

We have the following corollary by Theorems 4.2 and Corollary 4.9.

Corollary 4.10. *[2] If $f : X \to Y$ is a pseudo-open s-mapping of a metric space X onto a regular feebly compact space Y, then Y has a point-countable base.*

Theorem 4.11. *A regular space has a countable sp-network if and only if it is separable and has a point-countable sp-network.*

Proof. The "only if" part is trivial. To prove the "if" part, assume that a regular space X is separable and has a point-countable *sp*-network $\mathscr{P}$. Let D be a countable dense subset of X. We claim that the countable family

$$\mathscr{N} = \{\overline{P} : P \in \mathscr{P}, P \cap D \neq \varnothing\}$$

is an *sp*-network for X.

Fix any point $x \in X$, neighborhood $O_x \subset X$ of x and set $A \subset X$ with $x \in \overline{A}$. We need to find a set $N \in \mathscr{N}$ such that $x \in N \subset O_x$ and $x \in \overline{N \cap A}$. Since $x \in \overline{D}$ and X is a regular space, there is a set $P \in \mathscr{P}$ such that $x \in P \subset \overline{P} \subset O_x$ and $x \in \overline{P \cap D}$; thus $\overline{P} \in \mathscr{N}$. If $x \in A$, then $x \in \overline{P} \cap A \subset \overline{P} \subset O_x$. So we can assume that $x \notin A$. Since the space X has the strong Pytkeev property, it is of countable tightness and we can replace the set A by a smaller countable set and assume that $A = \{a_n\}_{n\in\omega} \subset X \setminus \{x\}$. Consider the (countable) subfamily $\mathscr{N}' = \{N \in \mathscr{N} : N \subset O_x\}$ and write it as $\mathscr{N}' = \{\overline{P}_k\}_{k\in\omega}$.

We claim that $x \in \overline{A \cap \overline{P}_k}$ for some $k \in \omega$. To derive a contradiction, assume that $x \notin \overline{A \cap \overline{P}_k}$ for all $k \in \omega$. Using the regularity of X, for every $k \in \omega$ fix an open neighborhood W_k of $\overline{A \cap \overline{P}_k}$ whose closure does not contain x.

For every $k \in \omega$ fix a neighborhood $U_k \subset X$ of a_k such that $x \notin \overline{U}_k$ and observe that

$$V_k := U_k \cap \bigcap\{W_i : i \leq k, a_k \in \overline{P}_i\} \setminus \bigcup\{\overline{P}_i : i \leq k, a_k \notin \overline{P}_i\}$$

is a neighborhood of a_k. Then the set

$$B := \bigcup_{k\in\omega} D \cap V_k$$

contains x in its closure. Consequently, there exists a set $P \in \mathscr{P}$ such that $\overline{P} \subset O_x$ and $x \in \overline{B \cap P}$. Since $B \subset D$, the set $\overline{P}$ belongs to $\mathscr{N}$ and $\mathscr{N}'$, so $P = P_k$ for some $k \in \omega$.

Let $\Omega = \{n \in \omega : a_n \in \overline{P}_k\}$ and observe that for every $n \geq k$ we have $V_n \subset W_k$ if $n \in \Omega$ and $V_n \cap \overline{P}_k = \varnothing$ if $n \notin \Omega$.

Finally, observe that

$$\begin{aligned} B \cap P_k = \bigcup_{n\in\omega} D \cap V_n \cap P_k &\subset \bigcup_{n\in\omega} V_n \cap \overline{P}_k \\ &\subset (\bigcup_{n<k} U_n) \cup (\bigcup_{k\leq n\in\Omega} V_n \cup \bigcup_{k\leq n\notin\Omega} V_n \cap \overline{P}_k) \\ &\subset (\bigcup_{n<k} U_n) \cup W_k \end{aligned}$$

and hence $x \notin \overline{B \cap P_k}$ as x does not belong to the closure of the set $(\bigcup_{n<k} U_n) \cup W_k$. □

Corollary 4.12. *Let X be a regular space having a point-countable sp-network consisting of separable subsets. Then X is a topological sum of strict $\mathfrak{P}_\circ$-spaces.*

Proof. Let $\mathscr{P}$ be a point-countable *sp*-network consisting of separable subsets for a topological space X. By Lemma 2.5(4), X is a meta-Lindelöf space. X is also a locally separable space, because for each $x \in X$, the set $\bigcup\{P \in \mathscr{P} : x \in P\}$ is a separable neighborhood of x (otherwise, $x \in \overline{X \setminus \bigcup\{P \in \mathscr{P} : x \in P\}}$, then there is a set $P_0 \in \mathscr{P}$ such that $x \in P_0$ and $x \in \overline{P_0 \cap (X \setminus \bigcup\{P \in \mathscr{P} : x \in P\})}$, which is a contradiction). By [25, Proposition 8.7], X is a topological sum of Lindelöf spaces. It is easy to see that every locally separable Lindelöf space is separable. Hence X is a topological sum of separable spaces. By Theorem 4.11, X is a topological sum of strict $\mathfrak{P}_\circ$-spaces. □

Corollary 4.13. *Assume that a regular space X has a point-countable sp-network. The space X is Fréchet-Urysohn at a point $x \in X$ if and only if X is discretely sequential at x.*

Proof. The "only if" part is trivial. To prove the "if" part, take any set $A \subset X$ and point $x \in \overline{A} \setminus A$. Since X has a countable *sp*-network at x, X is countably tight at x, so we can find a countable subset $B \subset A$ with $x \in \overline{B}$. The space $\overline{B}$ is separable and has a point-countable *sp*-network. By Theorem 4.11, $\overline{B}$ has a countable *sp*-network and hence is cosmic and hereditarily Lindelöf, which implies that $\overline{B}$ has countable closed pseudocharacter at x. By Theorem 3.4, $\overline{B}$ is Fréchet-Urysohn at x, which allows us to find a sequence of points $\{b_n\}_{n\in\omega} \subset B \subset A$ that converges to x. □

Remark 4.14. By Corollary 4.13, if a regular space X has a point-countable *sp*-network, then each k-subspace of X is Fréchet-Urysohn (see Theorem 4.2).

We recall that a topological space X has *countable cellularity* [13, p. 59] (*discrete cellularity*) if each pairwise disjoint (discrete) family of open sets in X is at most countable.

Question 4.15. Does every regular space have a countable *sp*-network if it has a point-countable *sp*-network and countable (discrete) cellularity?

Question 4.16. Find a characterization of spaces with a point-countable *sp*-network by certain images of metric spaces.

5. Spaces with σ-closure-preserving sp-networks

In this section, we discuss spaces with σ-closure-preserving *sp*-networks, prove that a topological space is a stratifiable space if and only if it is a regular space having a σ-closure-preserving *sp*-network, and every regular space with a σ-locally finite *sp*-network has a σ-discrete *sp*-network.

Let X be a topological space, and $\mathscr{P}$ be a family of subset of X. $\mathscr{P}$ is called *closure-preserving* [11, p. 350] if $\bigcup\{\overline{P} : P \in \mathscr{P}'\} = \overline{\bigcup\{P : P \in \mathscr{P}'\}}$ for each $\mathscr{P}' \subset \mathscr{P}$. $\mathscr{P}$ is called *discrete* [11, p. 349] if for each point x in X there is a neighborhood O_x of the point x such that the set O_x meets at most one element of the family $\mathscr{P}$. It is obvious that every discrete family of a space is locally finite; every locally finite family of a space is closure-preserving; and every disjoint and closure-preserving family of closed subsets of a space is discrete. Recall some spaces defined by σ-closure-preserving families. The space X is called an M_1*-space* [12, Definition 1.1] if it is a regular space with a σ-closure-preserving base. A family $\mathscr{B}$ of subsets of X is called a *quasi-base* [12, p. 105] for X if, for any neighborhood U of each point $x \in X$, there is $B \in \mathscr{B}$ such that $x \in B^\circ \subset B \subset U$. The space X is called an M_2*-space* [12, Definition 1.2] if it is a regular space with a σ-closure-preserving quasi-base. A family $\mathscr{P}$ of ordered pairs of subsets of X is called a *pair-base* [12, p. 106] for X if $\mathscr{P}$ satisfies the following conditions: (i) $(P_1, P_2) \in \mathscr{P} \Rightarrow P_1 \subset P_2$ and P_2 is open in X; (ii) for any neighborhood U of each point $x \in X$, there is $(P_1, P_2) \in \mathscr{P}$ such that $x \in P_1 \subset P_2 \subset U$. The space X is called an M_3*-space* [12, Definition 1.3] if it has a σ-cushioned pair-base, in which a family $\mathscr{F}$ of ordered pairs of subsets of X is called *cushioned* [11, p. 352] if $\overline{\bigcup\{P_1 : (P_1, P_2) \in \mathscr{F}'\}} \subset \bigcup\{P_2 : (P_1, P_2) \in \mathscr{F}'\}$ for each $\mathscr{F}' \subset \mathscr{F}$. The space is called a σ*-space* [23, Definition 4.3] if it is a regular space with a σ-closure-preserving (equivalently, σ-discrete) network [23, Theorem 4.11]. It is obvious that [23]

$$\text{metrizable spaces} \Rightarrow M_1\text{-spaces} \Rightarrow M_2\text{-spaces}$$
$$\Rightarrow M_3\text{-spaces} \Rightarrow \text{paracompact and } \sigma\text{-spaces.}$$

G. Gruenhage [22, Theorem 1] and H.J.K. Junnila [27, Theorem 4.17] independently proved that every M_3-space is an M_2-space. "Whether every M_2-space is an M_1-space" is one of the most difficult classic problems in general topology [24]. On the other hand, C.R. Borges introduced stratifiable spaces and proved that every stratifiable space is equivalent an M_3-space [10, Theorem 7.2].

Definition 5.1. A space X is called *semi-stratifiable* [31, Definition 1.4.3] if, there is a function G assigning an open set $G(n, F)$ for each $n \in \mathbb{N}$ and each closed set $F \subset X$ satisfying the following conditions:

(1) $F = \bigcap_{n \in \mathbb{N}} G(n, F)$;

(2) $F \subset K \Rightarrow G(n, F) \subset G(n, K)$ for each $n \in \mathbb{N}$.

If, in addition,

(3) $F = \bigcap_{n \in \mathbb{N}} \overline{G(n, F)}$,

then X is said to be *stratifiable* [10, Definition 1.1].

It is known that every stratifiable space is a σ-space, and every σ-space is semi-stratifiable [23, Theorem 5.9].

Definition 5.2. A space X is called *monotonically normal* [26, Definition 2.1] if, there is a function D assigning an open set $D(H, K)$ for each pair (H, K) of disjoint closed subsets of X satisfying the following conditions:

(1) $H \subset D(H, K) \subset \overline{D(H, K)} \subset X \setminus K$;

(2) if (H', K') is a pair of disjoint closed subsets of X, $H \subset H'$ and $K' \subset K$, then $D(H, K) \subset D(H', K')$.

The function D is called a *monotone normality operator* for X.

Lemma 5.3. *The following are equivalent for a space X:*

(1) X is a stratifiable space.

(2) X is a regular space with a σ-closure-preserving quasi-base [22,27].

(3) X is a monotonically normal, semi-stratifiable space [26, Theorem 2.5].

By *sp*-networks, a new characterization of stratifiable spaces is obtained, which gives a partial answer to Question 1.3.

Theorem 5.4. *A topological space is a stratifiable space if and only if it is a regular space with a σ-closure-preserving sp-network.*

Proof. It is easy to see that each quasi-base for a space is an *sp*-network. By Lemma 5.3, every stratifiable space is a regular space with a σ-closure-preserving *sp*-network.

Conversely, let X be a regular space with a σ-closure-preserving *sp*-network $\mathscr{F}$. By the regularity of X, the family $\mathscr{F}$ can be denoted by $\bigcup_{n\in\mathbb{N}}\mathscr{F}_n$, where each $\mathscr{F}_n$ is a closure-preserving family of closed subsets of X. It is easy to check that X is semi-stratifiable. By Lemma 5.3, we only need to prove that X is a monotonically normal space.

For each pair (H, K) of disjoint closed subsets of X, define $D(H, K) = (\bigcup_{n\in\mathbb{N}} U_n)^\circ$, where each

$$U_n = \bigcup\{F \in \bigcup_{i\leq n}\mathscr{F}_n : F\cap K = \varnothing\} \setminus \bigcup\{F \in \bigcup_{i\leq n}\mathscr{F}_n : F\cap H = \varnothing\}.$$

Next we will show that D is a monotone normality operator for X. First, if (H', K') is a pair of disjoint closed subsets of X, $H \subset H'$ and $K' \subset K$, then $D(H, K) \subset D(H', K')$. And then we prove that $H \subset D(H, K)$. If there is a point $x \in H \setminus D(H, K)$, then $x \in \overline{X \setminus \bigcup_{i\in\mathbb{N}} U_i} \cap H$. Because $x \in X \setminus K$, there exists a set $P \in \mathscr{F}$ such that $x \in P \subset X \setminus K$ and $x \in \overline{P \cap (X \setminus \bigcup_{i\in\mathbb{N}} U_i)}$. Suppose that $P \in \mathscr{F}_k$ for some $k \in \mathbb{N}$. Since $x \in H$ implies that $X \setminus \bigcup\{F \in \bigcup_{i\leq k}\mathscr{F}_i : F \cap H = \varnothing\}$ is an open neighborhood of the point x,

$$(X \setminus \bigcup\{F \in \bigcup_{i\leq k}\mathscr{F}_i : F\cap H = \varnothing\}) \cap P \cap (X \setminus \bigcup_{i\in\mathbb{N}} U_i) \neq \varnothing.$$

Since $P \setminus \bigcup\{F \in \bigcup_{i\leq k}\mathscr{F}_i : F \cap H = \varnothing\} \subset U_k$, $U_k \cap (X \setminus \bigcup_{i\in\mathbb{N}} U_i) \neq \varnothing$, which is a contradiction. Therefore, $H \subset D(H, K)$.

Finally, we prove that $\overline{D(H, K)} \subset X \setminus K$. Suppose that there is a point $x \in \overline{D(H, K)} \cap K$, then $x \in X \setminus H$. Because $\mathscr{F}$ is an *sp*-network for X, there exists a set $Q \in \mathscr{F}$ such that $x \in Q \subset X \setminus H$ and $x \in \overline{Q \cap D(H, K)}$. So we can choose $m \in \mathbb{N}$ such that $Q \in \mathscr{F}_m$, then $Q \cap U_k = \varnothing$ for each $k \geq m$. It shows that $x \in \overline{Q \cap \bigcup_{i<m} U_i} \subset \overline{\bigcup_{i<m} U_i} = \bigcup_{i<m}\overline{U_i}$. Therefore, there exists $i_0 < m$ such that

$$x \in \overline{U}_{i_0} \subset \bigcup\{F \in \bigcup_{i\leq i_0}\mathscr{F}_i : F \cap K = \varnothing\} \subset X \setminus K,$$

which is a contradiction. Thus we have that $\overline{D(H, K)} \subset X \setminus K$. It shows that X is a monotonically normal space. Hence, X is a stratifiable space. □

Remark 5.5. (1) It follows from Theorem 5.4 and Lemma 5.3 that every regular space with a σ-closure-preserving *sp*-network is a paracompact space, thus it is collectionwise normal [11, p. 352].

(2) There is a non-normal, sequential regular space with a σ-locally finite Pytkeev network (equivalently, k-network) [15, Example 3.3].

(3) Under Martin's Axiom and the negation of the continuum hypothesis, there is a k-and $\aleph_0$-space that is not monotonically normal (and hence not stratifiable) [15, Example 3.4].

The second part of this section, we discuss some spaces defined by σ-locally finite families. It is well known that a regular space has a σ-locally finite base if and only if it has a σ-discrete base [13, p. 282]. It is also true that a regular space has a σ-locally finite k-network (resp., cs^*-network, wcs^*-network) if and only if it has a σ-discrete k-network [14, Theorem 4] (resp., cs^*-network, wcs^*-network [31, Theorem 3.8.4]). The following result is obtained for *sp*-networks, which gives a partial answer to Question 1.2.

Let $\mathscr{P}$ and $\mathscr{Q}$ be the families of subsets of a set X. Define

$$\mathscr{P} \wedge \mathscr{Q} = \{P \cap Q : P \in \mathscr{P}, Q \in \mathscr{Q}\}.$$

The family $\mathscr{P}$ is called *star-finite* [11, p. 368] if each element of $\mathscr{P}$ only meets at most finitely many elements of $\mathscr{P}$.

Theorem 5.6. *The following are equivalent for a regular space X:*

(1) *X has a σ-discrete sp-network.*

(2) *X has a σ-locally finite sp-network.*

Proof. We only need to prove that (2) implies (1). Let $\mathscr{P} = \{P_\alpha : \alpha \in A\}$ be a σ-locally finite *sp*-network for the space X. By the regularity of X, we can assume that $\mathscr{P} = \bigcup_{m\in\mathbb{N}} \mathscr{P}_m$, where each $\mathscr{P}_m$ is a locally finite family of closed sets of X.

For each $m \in \mathbb{N}$, since $\mathscr{P}_m$ is locally finite in X, there exists an open cover $\mathscr{U}_m$ of X such that any element of $\mathscr{U}_m$ intersects at most finitely many elements of $\mathscr{P}_m$. By Theorem 5.4, X is a paracompact space, thus $\mathscr{U}_m$ has a σ-discrete closed refinement $\{F_\beta : \beta \in B_{m,n}, n \in \mathbb{N}\}$, where $\{F_\beta : \beta \in B_{m,n}\}$ is discrete for each $n \in \mathbb{N}$. It follows that, if $\beta \in \bigcup_{n\in\mathbb{N}} B_{m,n}$, then the set F_β only meets at most finitely many elements of $\mathscr{P}_m$.

By the paracompactness of X, for each pair $(m,n) \in \mathbb{N}^2$, there exists a pairwise disjoint family $\{W_\beta : \beta \in B_{m,n}\}$ of open subsets of X such that each $F_\beta \subset W_\beta$. Let

$$C_{m,n} = \{(\alpha,\beta) : P_\alpha \in \mathscr{P}_m, \beta \in B_{m,n} \text{ and } P_\alpha \cap F_\beta \neq \varnothing\}.$$

Then the family $\{P_\alpha \cap W_\beta : (\alpha,\beta) \in C_{m,n}\}$ is star-finite. Indeed, if $(P_\alpha \cap W_\beta) \cap (P_\gamma \cap W_\delta) \neq \varnothing$ for some $(\alpha,\beta), (\gamma,\delta) \in C_{m,n}$, the fact that $W_\beta \cap W_\delta \neq \varnothing$ and $\beta, \delta \in B_{m,n}$ forces $\beta = \delta$, and thus $(\gamma,\beta) \in C_{m,n}$, so $P_\gamma \cap F_\beta \neq \varnothing$. This shows that the set P_γ is one of the finitely many elements of $\mathscr{P}_m$ which meets the set F_β. So there are only finitely many pair $(\gamma,\delta) \in C_{m,n}$ for which $(P_\alpha \cap W_\beta) \cap (P_\gamma \cap W_\delta) \neq \varnothing$.

For each $(\alpha,\beta) \in C_{m,n}$ and each $i \in \mathbb{N}$, let

$$S(\alpha,\beta,i) = P_\alpha \cap \bigcup\{P_\gamma \in \mathscr{P}_i : P_\gamma \subset W_\beta\},$$

then $S(\alpha,\beta,i) \subset P_\alpha \cap W_\beta$. Define

$$\mathscr{S}(m,n,i) = \{S(\alpha,\beta,i) : (\alpha,\beta) \in C_{m,n}\}.$$

The family $\mathscr{S}(m,n,i)$ inherits the star-finite property from the family $\{P_\alpha \cap W_\beta : (\alpha,\beta) \in C_{m,n}\}$. Note too that each member of $\mathscr{S}(m,n,i)$ is the union of a subfamily of the locally finite family $\mathscr{P}_m \bigwedge \mathscr{P}_i$ and thus $\mathscr{S}(m,n,i)$ is closure-preserving. Because a star-finite family of sets is σ-disjoint [11, Lemma 3.10] and a disjoint and closure-preserving family of closed sets is discrete, the family $\mathscr{S}(m,n,i)$ is σ-discrete.

Define $\mathscr{S} = \bigcup\{\mathscr{S}(m,n,i) : m,n,i \in \mathbb{N}\}$. Then $\mathscr{S}$ is a σ-discrete family of closed subsets of X. We prove that $\mathscr{S}$ is an *sp*-network for X. Let $x \in U \cap \overline{Y}$ with U open and Y subset in X. Since $\mathscr{P}$ is an *sp*-network for X, there exist $m \in \mathbb{N}$ and $P_\alpha \in \mathscr{P}_m$ such that $x \in \overline{P_\alpha \cap Y}$ and $x \in P_\alpha \subset U$. Because $\bigcup_{n\in\mathbb{N}}\{F_\beta : \beta \in B_{m,n}\} = X$, there exist $n \in \mathbb{N}$ and $\beta \in B_{m,n}$ such that $x \in F_\beta$. Then $P_\alpha \cap F_\beta \neq \varnothing$, and $(\alpha,\beta) \in C_{m,n}$. By $x \in W_\beta$, there exist $i \in \mathbb{N}$ and $P_\gamma \in \mathscr{P}_i$ such that $x \in P_\gamma \subset W_\beta \cap U$ and $x \in \overline{P_\gamma \cap P_\alpha \cap Y}$. So we have that $x \in \overline{S(\alpha,\beta,i) \cap Y} \subset S(\alpha,\beta,i) \subset P_\alpha \subset U$. Thus, the family $\mathscr{S}$ is an *sp*-network for X. □

6. Topological groups with *sp*-networks

In this section, we will find some applications of *sp*-networks in topological spaces with algebra structures. It will be showed that every topological group with countable *sp*-character is metrizable.

Let G be a topological space with a group structure. The space G is called a *semitopological group* [5, p. 12] if the product map of $G \times G$ into G is separately continuous. The space G is called a *quasitopological group* [5, p. 12] if G is a semitopological group and the inverse map of G onto itself is continuous. The space G is called a *paratopological group* [5, p. 12] if the product map of $G \times G$ into G is jointly continuous. The space G is called a *topological group* [5, p. 12] if G is a paratopological group and the inverse map of G onto itself is continuous.

Some cardinal invariants are defined as follows. Let X be a topological space.

$$d(X) = \min\{|D| : \overline{D} = X\}, \text{ the } \textit{density} \text{ [13, P. 25] of } X;$$
$$nw(X) = \min\{|\mathscr{P}| : \mathscr{P} \text{ is a network for } X\}, \text{ the } \textit{network weight} \text{ [13, P. 127] of } X;$$
$$spnw(X) = \min\{|\mathscr{P}| : \mathscr{P} \text{ is an } sp\text{-network for } X\}, \text{ the } \textit{sp-network weight} \text{ of } X;$$
$$sp_\chi(X, x) = \min\{|\mathscr{P}| : \mathscr{P} \text{ is an } sp\text{-network at } x\}, \text{ the } \textit{sp-character} \text{ at } x \in X;$$
$$sp_\chi(X) = \sup\{sp_\chi(X, x) : x \in X\}, \text{ the } \textit{sp-character} \text{ of } X.$$

A topological space X is of *countable sp-character* if $sp_\chi(X) \le \omega$.

Theorem 6.1. *Let X be a paratopological group. Then $spnw(X) = nw(X)sp_\chi(X)$.*

Proof. It is obvious that $nw(X)sp_\chi(X) \le spnw(X)$. Next, we will show that $spnw(X) \le nw(X)sp_\chi(X)$. Assume that $\mathscr{P}$ is an *sp*-network at the unit e of the paratopological group X with $|\mathscr{P}| = sp_\chi(X, e)$. Let $\mathscr{N}$ be a network for X with $|\mathscr{N}| = nw(X)$.

We claim that the family $\mathscr{Q} = \{NP : N \in \mathscr{N}, P \in \mathscr{P}\}$ is an *sp*-network for the space X. Given a point $x \in U \cap \overline{A}$ with U open and A subset in X, we need to find sets $N \in \mathscr{N}$ and $P \in \mathscr{P}$ such that $x \in NP \subset U$ and $x \in \overline{NP \cap A}$. Because X is a paratopological group, we can find a neighborhood U_x of x and a neighborhood U_e of e in X such that $U_x U_e \subset U$. Since $\mathscr{N}$ is a network for X, there exists a set $N \in \mathscr{N}$ such that $x \in N \subset U_x$. Since $e \in \overline{x^{-1}A}$ and $\mathscr{P}$ is an *sp*-network at e, there exists a set $P \in \mathscr{P}$ such that $e \in P \subset U_e$ and $e \in \overline{P \cap x^{-1}A}$. Then $x \in NP \subset U_x U_e \subset U$ and $x \in x\overline{P \cap x^{-1}A} = \overline{xP \cap A} \subset \overline{NP \cap A}$. Therefore, $\mathscr{Q}$ is an *sp*-network for X. Since $|\mathscr{Q}| \le |\mathscr{N}||\mathscr{P}| = nw(X)sp_\chi(X, e)$, we conclude that $spnw(X) \le nw(X)sp_\chi(X)$. $\square$

Corollary 6.2. *If a paratopological group is a cosmic space with countable sp-character, then it has a countable sp-network.*

Remark 6.3. (1) Theorem 6.1 does not hold for quasitopological groups. There is a first-countable cosmic quasitopological group X which fails to be an $\aleph_0$-space [6, Example 4.11], and hence X does not have a countable *sp*-network.

(2) The result "$spnw(X) = d(X)sp_\chi(X)$" does not hold for paratopological groups X. Recall that the *Sorgenfrey line* is the reals $\mathbb{R}$ endowed with the topology generated by the base consisting of right half-open intervals $[a, b)$, $a < b$. It is a classical example of a paratopological group which is not a topological group [5, Example 1.2.1]. Let X be the Sorgenfrey line. Then X is a first-countable separable space and hence has countable *sp*-character. But X is not a cosmic space. It shows that X does not have a countable *sp*-network.

Theorem 6.4. *If a paratopological group X has countable sp-character, then either X is first-countable or $\{e\} = U \cap U^{-1}$ for some neighborhood $U \subset X$ of the unit e.*

Proof. Fix a countable *sp*-network $\mathscr{P}$ at the unit e of X. Enlarging $\mathscr{P}$ by a larger family, we can assume that $\mathscr{P}$ is closed under finite unions.

Assuming that $\{e\} \neq U \cap U^{-1}$ for every neighborhood $U \subset X$ of the unit e, we shall prove that the paratopological group X is first-countable. Consider the mapping $f : X \to X \times X$ defined by $f(x) = (x, x^{-1})$. By our assumption, the image $f(X)$ is a non-discrete subset in $X \times X$. Since (the topological homogeneous) space X has countable *sp*-character at e, the space X has the strong Pytkeev property. By [8, Theorem 3.1], the product $X \times X$ has the strong Pytkeev property and hence is countable tightness. Consequently, we can find a sequence of points $\{x_n\}_{n\in\omega}$ in $X \setminus \{e\}$ such that for any neighborhood $U \subset X$ of the unit e, there exists $n \in \omega$ such that $(x_n, x_n^{-1}) \in U \times U$.

Consider the countable family $\mathscr{N} = \{x_n^{-1}P : n \in \omega, P \in \mathscr{P}\}$ and let $\mathscr{N}^\circ = \{N^\circ : N \in \mathscr{N}, e \in N^\circ\}$ be the family of interiors N° of the sets $N \in \mathscr{N}$ that are neighborhoods of e. We claim that the countable family $\mathscr{N}^\circ$ is a neighborhood base at e.

Given any neighborhood $O_e \subset X$ of e, we should find $n \in \omega$ and $P \in \mathscr{P}$ such that $x_n^{-1}P$ is a neighborhood of e and $x_n^{-1}P \subset O_e$. By the continuity of the multiplication in the paratopological group X, there exists a neighborhood $U_e \subset X$ of e such that $U_eU_e \subset O_e$. Let $\Omega = \{n \in \omega : x_n, x_n^{-1} \in U_e\}$ and $\mathscr{P}' = \{P \in \mathscr{P} : P \subset U_e\}$. Since the family $\mathscr{P}$ is closed under finite unions, so is its subfamily $\mathscr{P}'$. It follows that $x_n^{-1}P \subset U_eU_e \subset O_e$ for any $n \in \Omega$ and $P \in \mathscr{P}'$.

We claim that for some $P \in \mathscr{P}'$ and $n \in \Omega$ the set $x_n^{-1}P$ is a neighborhood of e. To derive a contradiction, assume that $x_n^{-1}P$ is not a neighborhood of e for all $P \in \mathscr{P}'$ and $n \in \Omega$. Then P is not a neighborhood of x_n for all $P \in \mathscr{P}$ and $n \in \Omega$.

Write the countable family $\mathscr{P}'$ as $\{P_k\}_{k\in\omega}$. Since X is Hausdorff, each point x_n has a closed neighborhood W_n such that $e \notin W_n$. For every $n \in \Omega$ the set $\bigcup_{k\leq n} P_k$ belongs to $\mathscr{P}'$ and hence is not a neighborhood of x_n. Consequently, the set $W_n \setminus \bigcup_{k\leq n} P_k$ contains x_n in its closure and then the set

$$A = \bigcup_{n\in\Omega} (W_n \setminus \bigcup_{k\leq n} P_k)$$

contains e in its closure. Since $\mathscr{P}$ is an *sp*-network at e, there exists a set $P \in \mathscr{P}$ such that $P \subset U_e$ and $e \in \overline{A \cap P}$. Then $P \in \mathscr{P}'$ and hence $P = P_i$ for some $i \in \omega$. Now observe that

$$A \cap P = P_i \cap \bigcup_{n\in\Omega} (W_n \setminus \bigcup_{k\leq n} P_k) \subset P_i \cap \bigcup_{n<i} (W_n \setminus \bigcup_{k\leq n} P_k) \subset \bigcup_{n<i} W_n$$

and hence the unit e cannot belong to the closure of $A \cap P$ as e does not belong to the closure of the set $\bigcup_{n<i} W_n$. □

Corollary 6.5. *A topological group is metrizable if and only if it has countable sp-character.*

Proof. It is enough to prove the sufficiency. If a topological group X has countable *sp*-character, by Theorem 6.4, then the space X is first-countable; thus the topological group X is metrizable [5, Theorem 3.3.12]. □

Corollary 6.5 can be also derived from the following descriptions of topological spaces with countable *sp*-character (see Remark 6.7). Let $\mathscr{U}$ be a family of non-empty open subsets of a topological space X. $\mathscr{U}$ is called a π-*base* [5] at a point $x \in X$ if each neighborhood O_x of x contains some set $U \in \mathscr{U}$. $\mathscr{U}$ is called a π-*base* for X if $\mathscr{U}$ is a π-base at each point $x \in X$.

Theorem 6.6. *Assume that a topological space X has countable sp-character at a point $x \in X$. If one of the following conditions is satisfied, then X has a countable π-base at x.*

(1) *X is a regular space.*
(2) *X has countable tightness.*

Proof. (1) Assume that a regular space X has countable *sp*-character at x. Then X has countable tightness at x. So, we can find a countable set $C \subset X \setminus \{x\}$ containing x in its closure. By the regularity of X, there exists a decreasing sequence $\{U_n\}_{n\in\omega}$ of open neighborhoods of x such that $\overline{U}_{n+1} \subset U_n$ for every $n \in \omega$ and $C \cap \bigcap_{n\in\omega} U_n = \varnothing$. Let $U_\omega = \bigcap_{n\in\omega} U_n$. Then $U_\omega = \bigcap_{n\in\omega} \overline{U}_n$ is closed and observe that $x \in \overline{C} \cap U_0 \subset \overline{U_0 \setminus U_\omega}$.

Fix a countable *sp*-network $\mathscr{P}$ at x. Replacing each $P \in \mathscr{P}$ by its closure, we can assume that $\mathscr{P}$ consists of closed subsets of X. We claim that for any neighborhood $O_x \subset X$ of x, there exists a set $P \in \mathscr{P}$ such that $P \subset O_x$ and P has non-empty interior in X. To derive a contradiction, assume that for some neighborhood O_x of x each set $P \subset O_x$ with $P \in \mathscr{P}$ has empty interior and being closed and nowhere dense in X.

Consider the subfamily $\mathscr{P}' = \{P \in \mathscr{P} : P \subset O_x\}$ and write it as $\mathscr{P}' = \{P_i\}_{i\in\omega}$. For every $k \in \omega$, put

$$W_k = U_k \setminus (\overline{U}_{k+2} \cup \bigcup_{i\le k} P_i).$$

Since each P_k is nowhere dense in X, the set

$$\begin{aligned}\overline{W}_k &= \overline{(U_k \setminus \overline{U}_{k+2}) \cap \bigcap_{i\le k}(X \setminus P_i)} \\ &\supset (U_k \setminus \overline{U}_{k+2}) \cap \overline{X \setminus \bigcup_{i\le k} P_i} = U_k \setminus \overline{U}_{k+2}.\end{aligned}$$

Consequently, $U_0 \setminus U_\omega = \bigcup_{k\in\omega}(U_k \setminus \overline{U}_{k+2}) \subset \bigcup_{k\in\omega} \overline{W}_k$, thus the union $W = \bigcup_{k\in\omega} W_k$ is dense in $U_0 \setminus U_\omega$ and hence contains x in its closure. Since $\mathscr{P}'$ is an *sp*-network at x, there exists $k \in \omega$ such that $x \in \overline{P_k \cap W}$. Then also

$$x \in \overline{P_k \cap W \cap U_{k+1}} \subset \overline{P_k \cap (\bigcup_{i<k} W_i) \cap U_{k+1}} = \varnothing,$$

which is a desired contradiction.

This contradiction implies that the family $\mathscr{U}$ of non-empty interiors of the sets $P \in \mathscr{P}$ is a countable π-base at x.

(2) Assume that a topological space X have countable tightness at each point and a countable *sp*-network at a point $x \in X$. Let $\mathscr{P}$ be a countable *sp*-network at the point x. Put

$$\mathscr{B} = \{\bigcup \mathscr{P}' : \mathscr{P}' \in [\mathscr{P}]^{<\omega}\}.$$

Then $\mathscr{B}$ is countable. Next we will show that the family $\mathscr{B}^\circ = \{B^\circ : B \in \mathscr{B}\}$ is a π-base at x.

To derive a contradiction, assume that there exists an open neighborhood U of x such that U contains no element of $\mathscr{B}^\circ$. Because X has a countable tightness at the point x and $x \in \overline{U \setminus \{x\}}$, there exists a countable subset $A = \{x_i\}_{i\in\omega} \subset U \setminus \{x\}$ such that $x \in \overline{A}$. Write the family $\{P \in \mathscr{P} : x \in P \subset U\}$ as $\{P_i\}_{i\in\omega}$. For each $i \in \omega$, there is an open neighborhood U_i of the point x_i such that $U_i \subset U$ and $x \notin \overline{U}_i$. Since $(\bigcup_{j\le i} P_j)^\circ = \varnothing$, we have that $x_i \in U_i \cap \overline{X \setminus \bigcup_{j\le i} P_j} \subset \overline{U_i \setminus \bigcup_{j\le i} P_j}$. By the countable tightness of X, there exists a countable subset A_i of $U_i \setminus \bigcup_{j\le i} P_j$ such that $x_i \in \overline{A_i}$; thus $x \notin \overline{A_i}$ and $P_j \cap A_i = \varnothing$ for each $i > j$. Clearly, $x \in \overline{A} \subset \overline{\bigcup_{i\in\omega} A_i}$. Because $\mathscr{P}$ is a countable *sp*-network at the point x, there exists a set $P \in \mathscr{P}$ such that $x \in P \subset U$ and $x \in \overline{P \cap \bigcup_{i\in\omega} A_i}$. Thus we can choose $m \in \omega$ such that $P = P_m$. It shows that $x \in \overline{P \cap \bigcup_{i\le m} A_i} \subset \bigcup_{i\le m} \overline{A_i}$, which is a contradiction. □

Remark 6.7. The following is another proof of Corollary 6.5. If a topological group X has countable *sp*-character, then X has a countable π-base $\mathscr{U}$ at the unit e. Then the family $\{UU^{-1} : U \in \mathscr{U}\}$ is a countable neighborhood base at e. Thus X is metrizable.

Corollary 6.8. *If a topological space X has countable sp-character, then X is separable if and only if it has a countable π-base.*

Proof. The "only if" part is trivial. To prove the "if" part, assume that the space X is separable and has countable *sp*-character. Take any $x \in X$, by Theorem 6.6(2), X has a countable π-base $\mathscr{P}_x$ at x. Because X is separable, we can choose a countable subset $D = \{d_n\}_{n\in\omega}$ of X such that $\overline{D} = X$.

We claim that the countable family $\mathscr{P} = \bigcup\{\mathscr{P}_{d_n} : n \in \omega\}$ is a countable π-base for X. For each non-empty open subset U of X, there exists a point $d_n \in U$. Then we can choose a set $P \in \mathscr{P}_{d_n}$ such that $P \subset U$. It means that $\mathscr{P}$ is a countable π-base for X. □

The following example shows that the condition "*sp*-network" cannot be weakened to the "strict Pytkeev network" in Corollary 6.5, which gives a negative answer to Question 1.6.

By a *free topological group* [5, p. 409] over a topological space X we understand a pair $(F(X), i_X)$ consisting of a topological group $F(X)$ and a continuous map $i_X : X \to F(X)$ such that for every continuous map $f : X \to G$ to a topological group G there exists a unique continuous group homomorphism $h : F(X) \to G$ such that $f = h \circ i_X$.

Example 6.9. There is a non-Fréchet-Urysohn, sequential topological group with a countable strict Pytkeev network.

Proof. Let X be a copy of the sequential fan S_ω, see Example 4.3. Since X is a k_ω-space, it follows from [5, Theorem 7.4.1] that the free topological group $F(X)$ is also a k_ω-space, thus $F(X)$ is a k-space. Since X is an $\aleph_0$-space, $F(X)$ is an $\aleph_0$-space [4, Theorem 4.1]. By Lemmas 2.4(1) and 2.5(1), $F(X)$ is a sequential space with a countable Pytkeev network. Then $F(X)$ has a countable strict Pytkeev network. Since X is non-discrete, $F(X)$ is not first-countable [5, Theorem 7.1.20]. By Corollary 6.5 and Lemma 3.3, $F(X)$ is not Fréchet-Urysohn. □

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