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ON PERFECT IMAGES OF μ-SPACES

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Abstract. A space X is called a *μ-space* if it can be embedded in the product of countably many paracompact F_σ-metrizable spaces. K. Nagami in [15] posed the following problem: is the perfect image of a μ-space a μ-space?

By the saturated sets-topology of submetrizable spaces, in this paper the following theorem is proved, which gives a partial answer to Nagami's problem.

Theorem. *Let (X,τ) be a μ-space and $f:(X,\tau)\to(Y,\mathcal{U})$ a perfect mapping. Then*

(i) there are topologies $\{\tau_n\}_{n\in\omega}$ on X satisfying for each $n\in\mathbb{N}$ there is a saturated sets-topology $\mathcal{S}_n$ on (f,τ_n,τ_0) such that $\tau_0\subset\mathcal{S}_n\subset\tau$;

(ii) if $\mathcal{S}_n\subset\tau_n$ for each $n\in\mathbb{N}$, then $(Y,\mathcal{U})$ is a μ-space.

1. Introduction

M_i-spaces for $i=1,2$ and 3 were introduced by J. Ceder [1], which are important classes in generalized metric spaces [6, 10]. It is easy to see that every M_1-space is an M_2-space, and every M_2-space is an M_3-space. J. Ceder didn't know if any of these classes were in fact different. In the 1970s, G. Gruenhage [5] and H.J.K. Junnial [8] independently proved that M_3-spaces and M_2-spaces are the same. But to this day, it is not known if M_3-spaces and M_1-spaces are the

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same [13]. There are classes of spaces formally stronger than M_1-spaces for which it is as yet undetermined whether every M_3-space belongs to the classes [7]. The most pertinent of these classes is the class of μ-spaces, introduced by K. Nagami for dimension theory reasons [14, 15]. S. Oka [16] and T. Mizokami [11] showed that $\dim X = \mathrm{Ind} X$ for every μ-space X. Mizokami [12] proved every M_3-μ-space is M_1.

A space is called an *F_σ-metrizable* space if it is the union of countably many closed metrizable subspaces. A space X is called a *μ-space* in [14] if X can be embedded in the product of countably many paracompact F_σ-metrizable spaces. A mapping $f : X \to Y$ is called a *perfect* mapping if f is continuous closed onto and $f^{-1}(y)$ is compact for every $y \in Y$. Perfect mappings are a well-behaved class in terms of various mappings. The following interesting and long-standing difficult Nagami's problem [15] is still open.

Proposition 1.1. *Is the perfect image of a μ-space a μ-space?*

H.J.K. Junnila and T. Mizokami proved that the closed image of an M_3-F_σ-metrizable space is a μ-space [9], and K. Tamano [19] gave an example which is a continuous image of a separable metric space but not a μ-space. They gave a partial answer to Nagami's problem.

In this paper, we consider Nagami's problem, give it a partial answer. Let $f : X \to Y$ be a perfect mapping and X a μ-space, we can study the pre-image X instead of studying the image Y.

In this paper, all mappings are onto, all spaces are regular and T_1-spaces, and the letters $\mathbb{N}$, ω denote the set of positive integers, the set of natural numbers, respectively. For undefined notation and terminologies, the reader may refer to [4, 6].

2. Some lemmas and propositions

In this section, a characterization of μ-spaces is given by μ-bases, and a saturated sets-topology on a mapping is introduced, which will play an important role studying perfect images of μ-spaces. A topological space (X, τ) is called *submetrizable* [6] if there exists a metric ρ on X such that the metric topology τ_ρ induced by ρ is coarser than τ, and the metric ρ on X is called a *submetric* on X. A space is called a *σ-space* [6] if it has a σ-locally finite network, where a family $\mathcal{P}$ of subsets of a space X is called a *network* [4] for X if, whenever $x \in U$ with U open in X, there is $P \in \mathcal{P}$ such that $x \in P \subset U$. It is well-known that every

μ-space is a paracompact σ-space, and every paracompact σ-space is a submetrizable space [6]. V. Popov [17] gave an example which shows the perfect image of a hereditarily paracompact submetrizable space needs not be submetrizable.

If ρ is a metric on a set X, the metric topology on X induced by ρ is always denoted by τ_ρ in this paper. The following lemmas show that paracompact σ-spaces have special submetrics.

Lemma 2.1. [18, Lemma 2.20] *Let (X,τ) be a paracompact σ-space. If $\mathcal{D}$ is a σ-discrete family of open subsets of X, then there is a submetric d on X with $\mathcal{D} \subset \tau_d$.*

Lemma 2.2. [2, Theorem 2.8] *Let (X,τ_X) be a paracompact σ-space and $f : (X,\tau_X) \to (Y,\tau_Y)$ a perfect mapping. If ρ_0, d_0 are submetrics on spaces X, Y respectively, then there are a submetric metric ρ on (X,τ_X) and a submetric d on (Y,τ_Y) such that $f : (X,\rho) \to (Y,d)$ is a perfect mapping, $\tau_{\rho_0} \subset \tau_\rho$ and $\tau_{d_0} \subset \tau_d$.*

K. Tamano [19] showed the following result, which constructed some special metric and bases studying μ-spaces.

Theorem 2.3. [19, Lemma, p. 260] *A topological space (X,τ) is a μ-space if and only if there is an increasing sequence $\{\tau_n\}_{n\in\omega}$ of topologies on X satisfying the following conditions:*

(i) $\bigcup_{n\in\omega} \tau_n$ is a base of τ;

(ii) each (X,τ_n) is paracompact and (X,τ_0) is metrizable;

(iii) for every $n \in \mathbb{N}$, there is a sequence $\{X_{ni}\}_{i\in\mathbb{N}}$ of τ_0-closed sets of X such that $X = \bigcup_{i\in\mathbb{N}} X_{ni}$, and $\tau_n|_{X_{ni}} = \tau_0|_{X_{ni}}$ for each $i \in \mathbb{N}$.

For convenience's sake, a family $\bigcup_{n\in\omega} \tau_n$ of subsets of a μ-space (X,τ) is called a *μ-base* if it satisfies (i)-(iii) of Theorem 2.3. Let (X,τ) have a μ-base $\bigcup_{n\in\omega} \tau_n$. Then each (X,τ_n) is an F_σ-metrizable space by (ii) and (iii) of Theorem 2.3.

Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of a topological space X. Denote

$$\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Proposition 2.4. *Let $\bigcup_{n\in\omega} \tau_n$ be a μ-base for a μ-space (X,τ). If ρ is a submetric of (X,τ), then there exists a μ-base $\bigcup_{n\in\omega} \tau'_n$ for X such that $\tau_\rho \subset \tau'_0$ and $\tau_n \subset \tau'_n$ for each $n \in \omega$.*

Proof. Let $\mathcal{D}$ and $\mathcal{D}_0$ be σ-discrete bases of (X,τ_ρ) and (X,τ_0), respectively. Since (X,τ) is a paracompact σ-space, by Lemma 2.1, there is a submetric d on X such that $\mathcal{D} \cup \mathcal{D}_0 \subset \tau_d \subset \tau$.

Let $\tau'_0 = \tau_d$. Then $\tau_\rho \cup \tau_0 \subset \tau'_0$. For each $n \in \mathbb{N}$, let $\mathcal{B}_n = \tau_n \wedge \tau_d$. Then $\tau_n \cup \tau_d \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \subset \tau$, and $\mathcal{B}_n$ is a base for some topology τ'_n of X. Thus $\tau_n \cup \tau_d \subset \tau'_n \subset \tau'_{n+1} \subset \tau$ for each $n \in \omega$. By $\bigcup_{n\in\omega} \tau_n \subset \bigcup_{n\in\omega} \tau'_n \subset \tau$,

(i) $\bigcup_{n\in\mathbb{N}} \tau'_n$ *is a base of* τ.

(ii) *Each* (X, τ'_n) *is paracompact and* (X, τ'_0) *is metrizable.*

The space (X, τ'_n) is regular, because it is easy to see that $\mathrm{cl}_{\tau'_n}(U \cap V) \subset \mathrm{cl}_{\tau_n} U \cap \mathrm{cl}_{\tau_d} V$ for each $U, V \subset X$. To complete the proof it is enough to prove that every cover $\mathcal{O}$ of X by members of $\mathcal{B}_n$ has a σ-locally finite open refinement in (X, τ'_n). Let $\mathcal{P}$ be a σ-locally finite network for the F_σ-metrizable space (X, τ_n) and $\mathcal{E}$ a σ-locally finite base for the metrizable space (X, τ_d). Denote $\mathcal{O}$ by $\{U_\lambda \cap E_\lambda : U_\lambda \in \tau_n \text{ and } E_\lambda \in \tau_d \text{ for each } \lambda \in \Lambda\}$, and put

$$\begin{aligned}\mathcal{Q} =& \{P \cap E : P \in \mathcal{P}, E \in \mathcal{E}, P \subset U_\lambda \text{ and } E \subset E_\lambda \text{ for some } \lambda \in \Lambda\}\\ =& \{Q_\gamma : \gamma \in \Gamma\}\end{aligned}$$

Then $\mathcal{Q}$ is a cover of X, since $\mathcal{P}$ is a network for (X, τ_n) and $\mathcal{E}$ is a base for (X, τ_d). For each $\gamma \in \Gamma$, there are $P_\gamma \in \mathcal{P}$, $E_\gamma \in \mathcal{E}$ and $\lambda_\gamma \in \Lambda$ such that $Q_\gamma = P_\gamma \cap E_\gamma$, $P_\gamma \subset U_{\lambda_\gamma}$ and $E_\gamma \subset E_{\lambda_\gamma}$. It is well-known that if $\{F_s\}_{s\in S}$ is a locally finite family of subsets of a paracompact space, then there is a locally finite family $\{V_s\}_{s\in S}$ of open subsets such that $F_s \subset V_s$ for each $s \in S$ [4, Remark 5.1.19]. Since $\{P_\gamma : \gamma \in \Gamma\}$ is σ-locally finite in the paracompact space (X, τ_n), there is a σ-locally finite family $\{V_\gamma : \gamma \in \Gamma\}$ of open subsets in (X, τ_n) such that each $P_\gamma \subset V_\gamma \subset U_{\lambda_\gamma}$. Let $\mathcal{W} = \{V_\gamma \cap E_\gamma : \gamma \in \Gamma\}$. It can be checked that $\mathcal{W}$ is a σ-locally finite refinement of $\mathcal{O}$ in (X, τ'_n). Hence (X, τ'_n) is paracompact.

(iii) *For every* $n \in \mathbb{N}$, *there is a sequence* $\{X_{ni}\}_{i\in\mathbb{N}}$ *of* τ_0*-closed sets of* X *such that* $X = \bigcup_{i\in\mathbb{N}} X_{ni}$, *each* $\tau_n|_{X_{ni}} = \tau_0|_{X_{ni}}$ *and each* $\tau'_n|_{X_{ni}} = \tau'_0|_{X_{ni}}$.

In fact, $\tau_\rho \cup \tau_0 \subset \tau_d = \tau'_0$ by $\mathcal{D} \cup \mathcal{D}_0 \subset \tau_d$. It follows from (iii) of Theorem 2.3 that there is a sequence $\{X_{ni}\}_{i\in\mathbb{N}}$ of τ_0-closed sets of X such that $X = \bigcup_{i\in\mathbb{N}} X_{ni}$ and each $\tau_n|_{X_{ni}} = \tau_0|_{X_{ni}}$. Then each X_{ni} is a τ'_0-closed set of X.

We will prove $\tau'_n|_{X_{ni}} = \tau'_0|_{X_{ni}}$. It is obvious that $\tau'_0|_{X_{ni}} \subset \tau'_n|_{X_{ni}}$ by $\tau'_0 \subset \tau'_n$. Let $x \in O \cap X_{ni}$ with $O \in \tau'_n$. Since $\tau_n \wedge \tau_d$ is a base of τ'_n, there are $U \in \tau_n$ and $E \in \tau_d$ such that $x \in U \cap E \subset O$. Then $U \cap X_{ni} \in \tau_n|_{X_{ni}} = \tau_0|_{X_{ni}} \subset \tau_d|_{X_{ni}}$, $E \cap X_{ni} \in \tau_d|_{X_{ni}}$, and $x \in (U \cap X_{ni}) \cap (E \cap X_{ni}) \subset O \cap X_{ni}$. So $\tau'_n|_{X_{ni}} \subset \tau_d|_{X_{ni}} = \tau'_0|_{X_{ni}}$. □

Let $f : (X, \tau) \to (Y, \mathcal{U})$ be a mapping. A subset V of X is called a *saturated set* on the mapping f if $V = f^{-1}(f(V))$. For each $O \subset X$, define a set

$$S(O) = \bigcup\{f^{-1}(y) : y \in Y \text{ and } f^{-1}(y) \subset O\}.$$

The following can be checked easily that

(1) the set $S(O)$ is a saturated set on f;

(2) a subset V of X is saturated if and only if $V = S(V)$;

(3) a point $y \in f(S(O))$ if and only if $f^{-1}(y) \subset S(O)$;

(4) $S(O) = O \setminus f^{-1}(f(X \setminus O)) \subset O$.

Symbols $S(O)$ is used always with the same meaning throughout this paper. Next we introduce some properties of the set $S(O)$.

Proposition 2.5. *Assume $f : (X, \tau) \to (Y, \mathcal{U})$ be a mapping. Then*

(i) $S(O_1) \cap S(O_2) = S(O_1 \cap O_2)$ for each $O_1, O_2 \subset X$;

(ii) $S(\bigcup_{\alpha\in\Lambda} S(O_\alpha)) = \bigcup_{\alpha\in\Lambda} S(O_\alpha)$ for each $O_\alpha \subset X$;

(iii) $f(S(O_1)) \cap f(S(O_2)) = f(S(O_1) \cap S(O_2))$ for each $O_1, O_2 \subset X$;

(iv) $f(X \setminus S(O)) = Y \setminus f(S(O))$ for each $O \subset X$.

Proof. (i) Let $y \in Y$ and $f^{-1}(y) \subset S(O_1) \cap S(O_2)$. Then $f^{-1}(y) \subset O_1 \cap O_2$, thus $f^{-1}(y) \subset S(O_1 \cap O_2)$. Hence, $S(O_1) \cap S(O_2) \subset S(O_1 \cap O_2)$.

On the other hand, let $y \in Y$ and $f^{-1}(y) \subset S(O_1 \cap O_2)$. Then $f^{-1}(y) \subset O_1 \cap O_2$, thus $f^{-1}(y) \subset S(O_1) \cap S(O_2)$. Hence $S(O_1 \cap O_2) \subset S(O_1) \cap S(O_2)$.

(ii) It is clear that $S(\bigcup_{\alpha\in\Lambda} S(O_\alpha)) \subset \bigcup_{\alpha\in\Lambda} S(O_\alpha)$. Let $y \in Y$ and $f^{-1}(y) \subset \bigcup_{\alpha\in\Lambda} S(O_\alpha)$. Then $f^{-1}(y) \subset S(\bigcup_{\alpha\in\Lambda} S(O_\alpha))$ by the definition of $S(\bigcup_{\alpha\in\Lambda} S(O_\alpha))$. Thus $\bigcup_{\alpha\in\Lambda} S(O_\alpha) \subset S(\bigcup_{\alpha\in\Lambda} S(O_\alpha))$. Therefore, $S(\bigcup_{\alpha\in\Lambda} S(O_\alpha)) = \bigcup_{\alpha\in\Lambda} S(O_\alpha)$.

(iii) It is clear that $f(S(O_1) \cap S(O_2)) \subset f(S(O_1)) \cap f(S(O_2))$. On the other hand, let $y \in f(S(O_1)) \cap f(S(O_2))$. Then $f^{-1}(y) \subset S(O_1) \cap S(O_2)$, and $y \in f(S(O_1) \cap S(O_2))$. Thus $f(S(O_1)) \cap f(S(O_2)) \subset f(S(O_1) \cap S(O_2))$. Hence, $f(S(O_1)) \cap f(S(O_2)) = f(S(O_1) \cap S(O_2))$.

(iv) Since $S(O)$ is a saturated set on f, $X \setminus S(O) = X \setminus f^{-1}(f(S(O))) = f^{-1}(Y \setminus f(S(O)))$, thus $f(X \setminus S(O)) = Y \setminus f(S(O))$. □

Lemma 2.6. [3, Proposition 1][2, Lemma 2.2] *Let (X, τ) be a topological space and $f : (X, \tau) \to (Y, \mathcal{U})$ a continuous mapping. Then the following results are equivalent.*

(i) f is a closed mapping.

(ii) $\{S(O) : O \in \tau\} \subset \tau$ and $\mathcal{U} = \{f(S(O)) : O \in \tau\}$.

(iii) If $O \in \tau$, then $S(O) \in \tau$ and $f(S(O)) \in \mathcal{U}$.

Corollary 2.1. *Let $f : (X, \tau) \to (Y, \mathcal{U})$ be a closed continuous mapping. The family $\{S(O) : O \in \tau\}$ is a topology for X and the family $\{f(S(O)) : O \in \tau\}$ is a topology for Y.*

Proof. By (i) of Proposition 2.5, $\{S(O) : O \in \tau\}$ is closed under finite intersections. Let $O_\alpha \in \tau$ for each $\alpha \in \Lambda$. Then $\bigcup_{\alpha\in\Lambda} S(O_\alpha) = S(\bigcup_{\alpha\in\Lambda} S(O_\alpha))$ by (ii) of

Proposition 2.5, and $\bigcup_{\alpha\in\Lambda} S(O_\alpha) \in \tau$ by (iii) of Lemma 2.6. Thus $\{S(O) : O \in \tau\}$ is closed under unions. Hence, $\{S(O) : O \in \tau\}$ is a topology for X. In the same way, we can prove that $\{f(S(O)) : O \in \tau\}$ is also a topology for Y. □

In order to obtain non-trivial properties of the topology consisting of saturated sets on f, we need add some conditions related the mapping f and the topologies on X.

Proposition 2.7. *Let (X, τ) be a topological space and $f : (X, \tau) \to (Y, \mathcal{U})$ a continuous mapping. If τ_0 is a topology of X, then the family*

$$\mathcal{Q} = \{S(O) \cap V : O \in \tau \text{ and } V \in \tau_0\}$$

is a base of some topology for X.

Proof. Obviously, $X = S(X) \cap X \in \mathcal{Q}$. Let $O_1, O_2 \in \tau$, and $V_1, V_2 \in \tau_0$. By (i) of Proposition 2.5, we have $(S(O_1)\cap V_1)\cap(S(O_2)\cap V_2) = S(O_1\cap O_2)\cap(V_1\cap V_2) \in \mathcal{Q}$. Hence, $\mathcal{Q}$ is a base of some topology for X. □

The topology $\mathcal{S}$ generated by the base $\mathcal{Q}$ in Proposition 2.7 is called a *saturated sets-topology* [3, Definition 1] on (f, τ, τ_0). It is obvious that the topology $\mathcal{S}$ is generated by the subbase $\{S(O) : O \in \tau\} \cup \tau_0$. If $f : (X, \tau) \to (Y, \mathcal{U})$ is a continuous closed mapping and a topology τ_0 of X is coarser than τ, then the saturated sets-topology on (f, τ, τ_0) is coarser than τ by Lemma 2.6.

3. Main results

In this section, we discuss the perfect images of μ-spaces by μ-bases and saturated sets-topologies. The following result is a technical lemma.

Lemma 3.1. *Let (X, τ) be a μ-space and $f : (X, \tau) \to (Y, \mathcal{U})$ a perfect mapping. Then there is a μ-base $\bigcup_{n\in\omega} \tau_n$ for (X, τ) satisfying the following conditions:*

(i) for each $n \in \mathbb{N}$, let $\mathcal{S}_n$ be the saturated sets-topology on (f, τ_n, τ_0), then $\tau_0 \subset \mathcal{S}_n \subset \mathcal{S}_{n+1} \subset \tau$;

(ii) for each $n \in \mathbb{N}$, $f : (X, \mathcal{S}_n) \to (Y, \mathcal{U}_n)$ is a perfect mapping, where $\mathcal{U}_n = \{f(S(O)) : O \in \mathcal{S}_n\}$ is a topology of Y;

(iii) $f : (X, \mathcal{S}) \to (Y, \mathcal{U})$ is a perfect mapping, where topologies $\mathcal{S}$ and $\mathcal{U}$ are generated by bases $\bigcup_{n\in\mathbb{N}} \mathcal{S}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, respectively.

Proof. By Theorem 2.3, there exists a μ-base $\bigcup_{n\in\omega} \tau'_n$ for (X, τ). Since each μ-space is a paracompact σ-space, by Lemma 2.2, there are a submetric ρ on (X, τ) and a submetric d on $(Y, \mathcal{U})$ such that $f : (X, \rho) \to (Y, d)$ is a perfect mapping and $\tau_0' \subset \tau_\rho$. By Proposition 2.4, there exists a μ-base $\bigcup_{n\in\omega}\tau_n$ for (X,τ) such that $\tau_\rho \subset \tau_0$ and $\tau_n' \subset \tau_n$ for each $n \in \omega$. Thus $\tau_0' \subset \tau_\rho \subset \tau_0$.

For each $n \in \mathbb{N}$, let $\mathcal{E}_n = \{S(O) : O \in \tau_n\}$ and $\mathcal{S}_n$ the saturated sets-topology on (f, τ_n, τ_0) with a base $\mathcal{Q}_n = \mathcal{E}_n \wedge \tau_0$. By $\tau_0 \subset \tau_n \subset \tau_{n+1} \subset \tau$ and Lemma 2.6, it is obvious that $\tau_0 \subset \mathcal{Q}_n \subset \mathcal{S}_n \subset \mathcal{S}_{n+1} \subset \tau$. This completes the proof of (i).

Claim 1. For each $n \in \mathbb{N}$ and each $O \in \mathcal{S}_n$, $S(O) \in \mathcal{S}_n$.

In fact, let $x \in S(O)$ and $y = f(x)$. Then $x \in O$ and $f^{-1}(y) \subset S(O)$. If a point $t \in f^{-1}(y)$, then $t \in S(O) \subset O \in \mathcal{S}_n$; and there exist $O_t \in \tau_n$, $V_t \in \tau_0$ such that $t \in S(O_t) \cap V_t \subset O$ and $f^{-1}(y) \subset S(O_t)$. Let $\mathcal{Q}_x = \{S(O_t) \cap V_t : t \in f^{-1}(y)\}$. Then $\mathcal{Q}_x$ is a cover of the compact subset $f^{-1}(y)$ in (X, τ). Thus there exists a finite subfamily $\mathcal{Q}_x' = \{S(O_{t_i}) \cap V_{t_i} : i \leq m(x)\}$ of $\mathcal{Q}_x$, which covers $f^{-1}(y)$. Let $U_x = (\bigcap_{i\leq m(x)} O_{t_i}) \cap (\bigcup_{i\leq m(x)} V_{t_i})$. Then $f^{-1}(y) \subset U_x \in \tau_n$ by $\tau_0 \subset \tau_n$; thus $f^{-1}(y) \subset S(U_x) \in \mathcal{E}_n \subset \mathcal{S}_n$. By (i) of Proposition 2.5,

$$\begin{aligned} S(U_x) &= S(\bigcap_{i\leq m(x)} O_{t_i}) \cap S(\bigcup_{i\leq m(x)} V_{t_i}) \\ &= [\bigcap_{i\leq m(x)} S(O_{t_i})] \cap (\bigcup_{i\leq m(x)} V_{t_i}) \subset \bigcup_{i\leq m(x)} (S(O_{t_i}) \cap V_{t_i}) \subset O. \end{aligned}$$

Thus $x \in S(U_x) = S(S(U_x)) \subset S(O)$. Hence, $S(O) = \bigcup\{S(U_x) : x \in S(O)\} \in \mathcal{S}_n$. Claim 1 is proved.

By Claim 1 and the proof of Corollary 2.1, $\mathcal{U}_n = \{f(S(O)) : O \in \mathcal{S}_n\}$ is a topology for Y. We will show that $f : (X, \mathcal{S}_n) \to (Y, \mathcal{U}_n)$ is a perfect mapping. For each $O \in \mathcal{S}_n$, $f(S(O)) \in \mathcal{U}_n$ and $f^{-1}(f(S(O))) = S(O) \in \mathcal{S}_n$ by Claim 1. It follows from Lemma 2.6 that $f : (X, \mathcal{S}_n) \to (Y, \mathcal{U}_n)$ is continuous and closed. For each $y \in Y$, $f^{-1}(y)$ is compact in τ, thus it is compact in $\mathcal{S}_n$ because $\mathcal{S}_n \subset \tau$. In a word, $f : (X, \mathcal{S}_n) \to (Y, \mathcal{U}_n)$ is a perfect mapping. This completes the proof of (ii).

It is obvious that the family $\bigcup_{n\in\mathbb{N}} \mathcal{S}_n$ is closed under finite intersections. Thus $\bigcup_{n\in\mathbb{N}} \mathcal{S}_n$ is a base for some topology $\mathcal{S}$ of X. Let $\mathcal{E} = \{S(O) : O \in \tau\}$ and $\mathcal{S}^*$ the saturated sets-topology on (f, τ, τ_0) with a base $\mathcal{Q} = \mathcal{E} \wedge \tau_0$. Then $\mathcal{S}^* \subset \tau$ by Lemma 2.6.

Claim 2. $\mathcal{S} = \mathcal{S}^*$.

It is easy to see that $\mathcal{S} \subset \mathcal{S}^*$, because $\mathcal{E}_n \subset \mathcal{E}$ and $\mathcal{Q}_n \subset \mathcal{Q}$ for each $n \in \mathbb{N}$. To prove $\mathcal{S}^* \subset \mathcal{S}$, it is enough to prove $\mathcal{E} \subset \mathcal{S}$. Let $O \in \tau$, $x \in S(O)$ and $y = f(x)$. Then $x \in f^{-1}(y) \subset S(O) \in \mathcal{E} \subset \tau$. If a point $t \in f^{-1}(y)$, since $\bigcup_{n\in\mathbb{N}} \tau_n$ is a base of τ, there are $i(t) \in \mathbb{N}$ and $O_t \in \tau_{i(t)}$ with $t \in O_t \subset S(O)$. Then $\{O_t :$

$t \in f^{-1}(y)\}$ is an open cover of the compact subset $f^{-1}(y)$ in (X, τ), and there exists a finite subfamily $\{O_{t_j} : j \leq m(x)\}$ covering $f^{-1}(y)$. Let $U_x = \bigcup_{j \leq m(x)} O_{t_j}$ and $m = \max_{j \leq m(x)}\{i(t_j)\}$. Then $f^{-1}(y) \subset U_x \in \tau_m$ by each $\tau_i \subset \tau_{i+1}$. Thus $x \in S(U_x) \in \mathcal{E}_m \subset \mathcal{S}_m$ and $S(U_x) \subset U_x \subset S(O)$. Since $\bigcup_{n\in\mathbb{N}} \mathcal{S}_n$ is a base of $\mathcal{S}$, $\mathcal{S}^* \subset \mathcal{S}$. Claim 2 is proved.

By Claim 2 and Lemma 2.6, if $O \in \mathcal{S}$, then $S(O) \in \mathcal{S}$.

Claim 3. $\mathcal{U} = \mathcal{U}^*$, where $\mathcal{U}^* = \{f(S(O)) : O \in \mathcal{S}^*\}$.

In fact, if $U \in \mathcal{U}$, then $f^{-1}(U) \in \tau$. Thus $f^{-1}(U) = S(f^{-1}(U)) \in \mathcal{E} \subset \mathcal{S}^*$, and $U = f(S(f^{-1}(U))) \in \mathcal{U}^*$. Hence, $\mathcal{U} \subset \mathcal{U}^*$.

On the other hand, let $f(S(O)) \in \mathcal{U}^*$ with some $O \in \mathcal{S}^*$. It follows from $\mathcal{S}^* \subset \tau$ that $S(O) \in \mathcal{S}^*$, and $X \setminus S(O)$ is closed in (X, τ). Thus $f(X \setminus S(O))$ is closed in $(Y, \mathcal{U})$. By (iv) of Proposition 2.5, $f(X \setminus S(O)) = Y \setminus f(S(O))$. Then $Y \setminus f(S(O))$ is closed in $(Y, \mathcal{U})$, i.e., $f(S(O)) \in \mathcal{U}$. Hence, $\mathcal{U}^* \subset \mathcal{U}$. Claim 3 is proved.

Claim 4. $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ is a base of $\mathcal{U}$.

Let $U \in \mathcal{U}$. Then $U = f(S(O))$ with some $O \in \mathcal{S}^*$ by Claim 3. For every $y \in U$, there is an $x \in S(O)$ with $f(x) = y$, then $f^{-1}(y) \subset S(O) \in \mathcal{S}^*$. By Claim 2, the family $\bigcup_{n\in\mathbb{N}} \mathcal{S}_n$ is a base of $\mathcal{S}^*$. If a point $t \in f^{-1}(y)$, there exists $O_t \in \mathcal{S}_n$ for some $n \in \mathbb{N}$ with $t \in O_t \subset S(O)$. Let $\mathcal{O} = \{O_t : t \in f^{-1}(y)\}$. Then $\mathcal{O} = \{O_t : t \in f^{-1}(y)\}$ is an open cover of the compact subset $f^{-1}(y)$ in (X, τ), and there is a finite subfamily $\mathcal{O}'$ of $\mathcal{O}$ covering $f^{-1}(y)$. Let $U_x = \cup\mathcal{O}'$. Then $f^{-1}(y) \subset U_x \subset S(O)$, and by each $\mathcal{S}_n \subset \mathcal{S}_{n+1}$, $U_x \in \mathcal{S}_m$ for some $m \in \mathbb{N}$. Thus $f^{-1}(y) \subset S(U_x)$ and $f(S(U_x)) \in \mathcal{U}_m \subset \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$. Therefore, $y \in f(S(U_x)) \subset f(S(O)) = U$. Hence, $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ is a base of $\mathcal{U}$. Claim 4 is proved.

Next, we prove that $f : (X, \mathcal{S}) \to (Y, \mathcal{U})$ is a perfect mapping. For each $U \in \mathcal{U}$, $U = f(S(O))$ for some $O \in \mathcal{S}^*$ by Claim 3; and $f^{-1}(U) = S(O) \in \mathcal{S}^*$. Thus $f : (X, \mathcal{S}) \to (Y, \mathcal{U})$ is continuous. Let $W \in \mathcal{S}$. By Claims 2, 3 and Lemma 2.6, $S(W) \in \mathcal{S}$ and $f(S(W)) \in \mathcal{U}$. It follows from Lemma 2.6 that $f : (X, \mathcal{S}) \to (Y, \mathcal{U})$ is a closed mapping. For every $y \in Y$, $f^{-1}(y)$ is compact in (X, τ). Since $\mathcal{S} \subset \tau$, $f^{-1}(y)$ is compact in $(X, \mathcal{S})$. In a word, $f : (X, \mathcal{S}) \to (Y, \mathcal{U})$ is a perfect mapping. This completes the proof of (iii). □

The μ-base $\bigcup_{n\in\omega} \tau_n$ for (X, τ) in Lemma 3.1 is called a *special μ-base* on (f, τ). The following theorem gives a partial answer to Nagami's problem.

Theorem 3.2. *Let $f:(X,\tau)\to(Y,\mathcal{U})$ be a perfect mapping and (X,τ) a μ-space with a special μ-base $\bigcup_{n\in\omega}\tau_n$ on (f,τ). Then $(Y,\mathcal{U})$ is a μ-space, if $S(O)\in\tau_n$ for each $n\in\mathbb{N}$ and each $O\in\tau_n$.*

Proof. For the simplicity of the proof, we use notation in the proof of Lemma 3.1. It is easy to check that for each $n\in\mathbb{N}$, $\mathcal{S}_n\subset\tau_n$ if and only if $S(O)\in\tau_n$ for each $O\in\tau_n$. Let $\mathcal{U}_n^\tau=\{f(S(O)):O\in\tau_n\}$. By the proof of Corollary 2.1, $\mathcal{U}_n^\tau$ is a topology for Y. Let $\mathcal{U}_0^\tau=\{f(S(O)):O\in\tau_\rho\}$.

Claim 5. $\tau_d=\mathcal{U}_0^\tau\subset\mathcal{U}_n\subset\mathcal{U}_n^\tau\subset\mathcal{U}_{n+1}^\tau\subset\mathcal{U}$ for each $n\in\mathbb{N}$.

In fact, $\mathcal{U}_0^\tau\subset\mathcal{U}_n$ by $\tau_\rho\subset\tau_0\subset\mathcal{S}_n$. By Lemma 2.6, $\mathcal{U}_n\subset\mathcal{U}_n^\tau\subset\mathcal{U}_{n+1}^\tau\subset\mathcal{U}$. Let $f(S(O))\in\mathcal{U}_0^\tau$ with some $O\in\tau_\rho$. Since $f:(X,\rho)\to(Y,d)$ is a perfect mapping, it follows from (iii) of Lemma 2.6 that $f(S(O))\in\tau_d$. Then $\mathcal{U}_0^\tau\subset\tau_d$. On the other hand, if $V\in\tau_d$, then $f^{-1}(V)\in\tau_\rho$, and $V=f(f^{-1}(V))=f[S(f^{-1}(V))]\in\mathcal{U}_0^\tau$. Thus $\tau_d\subset\mathcal{U}_0^\tau$. Claim 5 is proved.

Claim 6. $(Y,\mathcal{U}_n^\tau)$ is a paracompact F_σ-metrizable space.

For each $O\in\tau_n$, $f^{-1}(f(S(O)))=S(O)\in\mathcal{S}_n\subset\tau_n$ and $f(S(O))\in\mathcal{U}_n^\tau$. Then $f:(X,\tau_n)\to(Y,\mathcal{U}_n^\tau)$ is continuous and closed by Lemma 2.6. For each $y\in Y$, $f^{-1}(y)$ is compact in τ, thus it is compact in τ_n. This shows that $f:(X,\tau_n)\to(Y,\mathcal{U}_n^\tau)$ is a perfect mapping. Since (X,τ_n) is paracompact, $(Y,\mathcal{U}_n^\tau)$ is also paracompact. By (iii) of the proof of Proposition 2.4, let $X=\bigcup_{i\in\mathbb{N}}X_{ni}$, where each X_{ni} is a τ_0'-closed set, and each $\tau_n|_{X_{ni}}=\tau_0|_{X_{ni}}$. Let $Y_{ni}=f(X_{ni})$. It is obvious that $Y=\bigcup_{i\in\mathbb{N}}Y_{ni}$. Since each X_{ni} is τ_0'-closed, X_{ni} is τ_ρ-closed by $\tau_0'\subset\tau_\rho$. So Y_{ni} is τ_d-closed, and Y_{ni} is $\mathcal{U}_n^\tau$-closed by $\tau_d\subset\mathcal{U}_n^\tau$. Since X_{ni} is τ_n-closed, $f|_{X_{ni}}:(X_{ni},\tau_n|_{X_{ni}})\to(Y_{ni},\mathcal{U}_n^\tau|_{Y_{ni}})$ is a perfect mapping. Since $\tau_n|_{X_{ni}}=\tau_0|_{X_{ni}}$ and (X,τ_0) is metrizable, $(X_{ni},\tau_n|_{X_{ni}})$ and $(Y_{ni},\mathcal{U}_n^\tau|_{Y_{ni}})$ are metrizable subspaces. Claim 6 is proved.

Next, we show that $(Y,\mathcal{U})$ is a μ-space. For each $n\in\mathbb{N}$, let $Y_n=(Y,\mathcal{U}_n^\tau)$, and $\mathrm{id}_n:(Y,\mathcal{U})\to Y_n$ be the identity mapping. Then Y_n is a paracompact F_σ-metrizable space by Claim 6; and id_n is continuous by Claim 5. If a subset $A\subset Y$ is closed in $(Y,\mathcal{U})$ and a point $y\in Y\setminus A$, by Claims 4 in the proof of Lemma 3.1 and 5, there are $n\in\mathbb{N}$ and $U\in\mathcal{U}_n^\tau$ such that $y\in U\subset Y\setminus A$, thus $\mathrm{cl}_{Y_n}(A)\subset Y\setminus U$, and $y\notin\mathrm{cl}_{Y_n}(A)=\mathrm{cl}_{Y_n}(\mathrm{id}_n(A))$. This shows the family $\{\mathrm{id}_n\}_{n\in\mathbb{N}}$ of continuous mappings separates points from closed sets in $(Y,\mathcal{U})$. A mapping $g:(Y,\mathcal{U})\to\prod_{n\in\mathbb{N}}Y_n$ is defined by $g(y)=(\mathrm{id}_n(y))_{n\in\mathbb{N}}$. By the diagonal theorem [4, Theorem 2.3.20], the mapping g is an embedding mapping. Hence, $(Y,\mathcal{U})$ is a μ-space. □

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