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On statistical convergence in cone metric spaces[☆]

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ABSTRACT

Every metric space is a cone metric space, and every cone metric space is a topological space. In this paper, we introduce and investigate statistical convergence in cone metric spaces, discuss statistically-sequentially compact spaces and characterize statistical completeness of cone metric spaces.

1. Introduction and preliminaries

The notion of statistical convergence of sequences in real spaces was introduced in 1951 by H. Fast in [8] and H. Steinhaus in [19], respectively. This concept has been studied by many mathematicians up to date (see, for example [4,9,10,14]). Recently, the notion of statistical convergence was defined in topological spaces by G. Di Maio and Lj.D.R. Kočinac in [15].

In this paper we shall study statistical convergence in cone metric spaces. Cone metric spaces have been actually defined many years ago by several authors and appeared in the literature under different names. Dj. Kurepa was the first who introduced such spaces in 1934 under the name "espaces pseudo-distenciés" [16].

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Cone metric spaces are one of many generalizations of metric spaces, and play an important role in fixed point theory, computer science, and some other research areas as well as in general topology (see, for example [1,2,5,6,17]). There have been a lot of papers dealing with the theory of cone metric spaces. Recently, the paracompactness and the metrizability of cone metric spaces have been discussed in [18] and [13], respectively. In this paper, we shall introduce and investigate statistical convergence in cone metric spaces, discuss statistically-sequentially compact spaces and characterize statistical completeness of cone metric spaces. The paper is organized so that introduction is followed by three sections. In Section 2 we familiarize the reader with the basic notions concerning statistical convergence in cone metric spaces and give some topological natures of this convergence. In Section 3 we apply idea of statistical convergence to define statistically-sequential compactness and statistical completeness of cone metric spaces and give an explicit characterization.

Throughout this paper, the set of positive integers is denoted by $\mathbb{N}$, the set of real numbers with the standard topology is denoted by $\mathbb{R}$. For undefined terms in the paper the readers can refer to [7,11].

Definition 1.1. ([3]) Let E be a real Banach space and P a subset of E. We call P a *cone* and (E, P) a *cone space* if

(C1) P is non-empty, closed, and $P \neq \{0\}$;
(C2) $0 \leqslant a, b \in \mathbb{R}$ and $x, y \in P \Longrightarrow ax + by \in P$;
(C3) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

A partial ordering "$\leqslant$" with respect to P is defined by $x \leqslant y \Longleftrightarrow y - x \in P$, and $x < y \Longleftrightarrow x \leqslant y$ and $x \neq y$.

$x \ll y$ indicates $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P (with the topology of the Banach space E). The relation "$\ll$" is transitive and antisymmetric but not in general reflective. In this paper, we always assume that $\text{int}P \neq \emptyset$, and denote $E^+ = \{c \in E : 0 \ll c\}$, i.e., $E^+ = \text{int}P$.

Let $c \in E^+$ and $e \in E$. If $\{a_n\}$ is a non-negative sequence in $\mathbb{R}$ such that it converges to 0. It is clear that the sequence $\{c - a_n e\}$ in E converges to $c \in E^+$. So there is $n \in \mathbb{N}$ such that $c - a_n e \in E^+$, i.e., $0 \ll c - a_n e$. It follows that $a_n e \ll c$ for some $n \in \mathbb{N}$.

Definition 1.2. ([12]) Let (E, P) be a cone space, X a non-empty set and $d : X \times X \to E$ a mapping that satisfies the following conditions:

(CM1) $0 \leqslant d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Longleftrightarrow x = y$;
(CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(CM3) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X and (X, E, P, d) (or shortly, (X, d)) a *cone metric space*.

The notion of cone metric spaces are generated by replacing positive real numbers with a positive cone in a Banach space. It is obvious that every metric space is a cone metric space. Every cone metric space (X, d) is a topological space [20]. In fact, for any $c \in E^+$, let $B(x, c) = \{y \in X : d(x, y) \ll c\}$ (a c-ball in a cone metric space). Then

$$\mathcal{B} = \{B(x, c) : x \in X, c \in E^+\}$$

is a base of a topology $\tau_d = \{U \subset X : \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subset U\}$ on X. It can be shown that the topology τ_d is Hausdorff and first countable [20].

Definition 1.3. ([8]) Let $A \subset \mathbb{N}$, put $A(n) = \{k \in A : k \leqslant n\}$, $\forall n \in \mathbb{N}$. Then

$$\underline{\delta}(A) = \lim_{n\to\infty} \inf \frac{|A(n)|}{n} \qquad \text{and} \qquad \bar{\delta}(A) = \lim_{n\to\infty} \sup \frac{|A(n)|}{n}$$

are called *lower and upper asymptotic density* of the set A, respectively.

If $\underline{\delta}(A) = \bar{\delta}(A)$, then

$$\delta(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}$$

is called an *asymptotic (or natural) density* of the set A. All the three densities, if they exist, are in [0, 1].

A subset A of $\mathbb{N}$ is said to be *statistically dense* if $\delta(A) = 1$.

It is easy to see that $\delta(\mathbb{N} - A) = 1 - \delta(A)$ for each $A \subset \mathbb{N}$.

Definition 1.4. ([8]) A sequence $\{x_n\}$ in $\mathbb{R}$ is said to be *statistically convergent* to a point $x \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n : |x_k - x| \geqslant \varepsilon\}| = 0, \text{i.e.,} \lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n : |x_k - x| < \varepsilon\}| = 1.$$

2. Statistical convergence in cone metric spaces

In this section, we recall the concept of convergence of sequences in cone metric spaces, introduce the concept of statistical convergence of sequences in cone metric spaces, and present some basic results.

Definition 2.1. ([12]) Let (X, d) be a cone metric space.

(1) Let $\{x_n\}$ be a sequence in X and $x \in X$. If for each $c \in E^+$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ converges to x.
(2) Let $\{x_n\}$ be a sequence in X. If for each $c \in E^+$, there is $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a *Cauchy sequence* in X.
(3) (X, d) is said to be *complete* if every Cauchy sequence in X is convergent in X.

According to the sequential convergence and completeness of metric spaces, we give the following definitions about cone metric spaces.

Definition 2.2. Let (X, d) be a cone metric space, and $\{x_n\}$ a sequence in X. Then

(1) $\{x_n\}$ is said to be *statistically convergent* to a point $x \in X$, if for each $c \in E^+$, we have $\delta\{n \in \mathbb{N} : d(x_n, x) \leqslant c\} = 1$. This is denoted by *st*-$\lim\limits_{n\to\infty} x_n = x$.
(2) $\{x_n\}$ is called a *statistical Cauchy sequence* in X, if for each $c \in E^+$, there is $n_0 \in \mathbb{N}$ such that $\delta\{n \in \mathbb{N} : d(x_n, x_{n_0}) \leqslant c\} = 1$.
(3) (X, d) is said to be *statistically complete* if every statistical Cauchy sequence in X is statistically convergent.

The statistical convergence of sequences in a cone metric space is a natural generalization of the usual convergence. Let $\{x_n\}$ be a sequence in a cone metric space (X, d). If $\lim\limits_{n\to\infty} x_n = x \in X$, then for each $c \in E^+$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \leqslant c, \forall n > n_0$. Thus $\forall n > n_0$,

$$|A(n)| = |\{k \leqslant n : d(x_k, x) \leqslant c\}| \geqslant n - n_0,$$

and

$$\lim_{n\to\infty} \frac{|A(n)|}{n} = 1.$$

Hence $st\text{-}\lim_{n\to\infty} x_n = x$. Therefore, every convergent sequence is statistical convergent in a cone metric space. The converse is not true.

Example 2.3. ([12]) Let $E = \mathbb{R}^2$ with the usual Euclidean norm, $P = \{(x, y) \in E : x, y \geqslant 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha > 0$ is a constant. Then (X, d) is a cone metric space.

A sequence $\{x_n\}$ in X is defined by

$$x_n = \begin{cases} 1/n, & n \neq m^2, \\ n, & n = m^2, m \in \mathbb{N}. \end{cases}$$

For $x = 0$, we have that $d(x_n, x) = (1/n, \alpha/n)$ if $n \neq m^2$ and $d(x_n, x) = (n, \alpha n)$ if $n = m^2$, and $m \in \mathbb{N}$. For each $c \in E^+$, we have

$$A(n) = \{k \leqslant n : d(x_k, x) \leqslant c\} \supset \{k \leqslant n : k > n_c, k \neq m^2, m \in \mathbb{N}\}$$

for some $n_c \in \mathbb{N}$. Then

$$\delta\{n \in \mathbb{N} : d(x_n, x) \leqslant c\} \geqslant \delta\{n \in \mathbb{N} : n > n_c, n \neq m^2, m \in \mathbb{N}\} = 1.$$

Consequently, $st\text{-}\lim_{n\to\infty} x_n = x$, but the sequence $\{x_n\}$ is not convergent.

Example 2.4. Let $E = \mathbb{R}^2$ with the usual Euclidean norm, $P = \{(x, y) \in E : x, y \geqslant 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined by

$$d(x, y) = \left(\left|\frac{x}{1 + |x|} - \frac{y}{1 + |y|}\right|, \sqrt{3}\left|\frac{x}{1 + |x|} - \frac{y}{1 + |y|}\right|\right).$$

It can easily verify that (X, d) is a cone metric space.

A sequence $\{x_n\}$ in X is defined by

$$x_n = \begin{cases} n, & n \neq m^3, \\ 1/n, & n = m^3, m \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}$ is a statistical Cauchy sequence in X, but it is not statistically convergent.

In fact, for each $c \in E^+$, there exists $n_0 \in \mathbb{N}$ such that $n_0 \neq m^3$ for each $m \in \mathbb{N}$ and $2/n_0 < \|c\|$. If $n > n_0$ and $n \neq m^3$ for each $m \in \mathbb{N}$, then

$$\begin{aligned} \|d(x_n, x_{n_0})\| &= \left\|\left(\left|\frac{n}{1+n} - \frac{n_0}{1+n_0}\right|, \sqrt{3}\left|\frac{n}{1+n} - \frac{n_0}{1+n_0}\right|\right)\right\| \\ &= 2\frac{n - n_0}{(1+n)(1+n_0)} < \frac{2}{n_0}. \end{aligned}$$

Thus

$$A(n)=\{k\leqslant n: d(x_k,x_{n_0})\leqslant c\}\supset\{k\leqslant n: k>n_c, k\neq m^3, m\in\mathbb{N}\}$$

for some $n_c\in\mathbb{N}$. Then

$$\delta\{n\in\mathbb{N}: d(x_n,x_{n_0})\leqslant c\}\geqslant\delta\{n\in\mathbb{N}: n>n_c, n\neq m^3, m\in\mathbb{N}\}=1.$$

Hence $\{x_n\}$ is a statistical Cauchy sequence in X.

If $a\in\mathbb{R}$ and $n\in\mathbb{N}$ with $n\neq m^3$ for each $m\in\mathbb{N}$, then

$$\|d(x_n,a)\|=\|(|\frac{n}{1+n}-\frac{a}{1+|a|}, \sqrt{3}|\frac{n}{1+n}-\frac{a}{1+|a|}|)\|=\frac{2|n-a|}{(1+n)(1+|a|)}.$$

Since $\lim\limits_{n\to\infty}\frac{2|n-a|}{(1+n)(1+|a|)}=\frac{2}{1+|a|}\neq 0$, $\{x_n\}$ is not a statistically convergent sequence.

The following results can be easily showed by the definitions of statistical convergence of cone metric spaces.

Lemma 2.5. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a cone metric space (X,d).*

(1) *If $st\text{-}\lim\limits_{n\to\infty}x_n=x$ and $st\text{-}\lim\limits_{n\to\infty}x_n=x_1$, then $x=x_1$.*
(2) *$st\text{-}\lim\limits_{n\to\infty}x_n=x$ if and only if $st\text{-}\lim\limits_{n\to\infty}d(x_n,x)=0$.*
(3) *If $st\text{-}\lim\limits_{n\to\infty}x_n=x$ and $st\text{-}\lim\limits_{n\to\infty}y_n=y$, then $st\text{-}\lim\limits_{n\to\infty}d(x_n,y_n)=d(x,y)$.*

Theorem 2.6. *Let $\{x_n\}$ and $\{z_n\}$ be two sequences in a cone metric space (X,d). If $st\text{-}\lim\limits_{n\to\infty}z_n=a$, and $d(x_n,a)\leqslant d(z_n,a)$ for each $n\in\mathbb{N}$, then $st\text{-}\lim\limits_{n\to\infty}x_n=a$.*

Proof. Since $st\text{-}\lim\limits_{n\to\infty}z_n=a$, $st\text{-}\lim\limits_{n\to\infty}d(z_n,a)=0$ by Lemma 2.5(2), for each $c\in E^+$ and $n\in\mathbb{N}$, we have

$$\{k\leqslant n: d(x_k,a)\leqslant c\}\supset\{k\leqslant n: d(z_k,a)\leqslant c\},$$

and

$$\delta\{n\in\mathbb{N}: d(x_n,a)\leqslant c\}\geqslant\delta\{n\in\mathbb{N}: d(z_n,a)\leqslant c\}=1.$$

Consequently, $st\text{-}\lim\limits_{n\to\infty}d(x_n,a)=0$, i.e., $st\text{-}\lim\limits_{n\to\infty}x_n=a$. □

Definition 2.7. ([15]) A subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$ is *statistically dense* in $\{x_n\}$ if the index set $\{n_k: k\in\mathbb{N}\}$ is a statistically dense subset of $\mathbb{N}$, i.e., $\delta\{n_k: k\in\mathbb{N}\}=1$.

Theorem 2.8. *Let $\{x_n\}$ be a sequence in a cone metric space (X,d). Then the following are equivalent:*

(1) *$\{x_n\}$ is statistically convergent in (X,d);*
(2) *there is a convergent sequence $\{y_n\}$ in X such that $x_n=y_n$ for almost all $n\in\mathbb{N}$;*
(3) *there is a statistically dense subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that the sequence $\{x_{n_k}\}$ is convergent;*
(4) *there is a statistically dense subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that the sequence $\{x_{n_k}\}$ is statistically convergent.*

Proof. (1) ⇒ (2). Suppose that $st\text{-}\lim_{n\to\infty} x_n = a \in X$. For each $c \in E^+$, we have

$$\delta\{n \in \mathbb{N} : d(x_n, a) \leqslant c\} = \lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n : d(x_k, a) \leqslant c\}| = 1.$$

Choose $e \in E$ with $\|e\| = 1$. Since $\lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n : d(x_k, a) \leqslant e/2\}| = 1$, there exists $n_1 \in \mathbb{N}$ such that $\frac{1}{n}|\{k \leqslant n : d(x_k, a) \leqslant e/2\}| > 1 - 1/2$ for each $n > n_1$. Choose an increasing sequence $\{n_k\}$ of positive integers such that $\frac{1}{n}|\{m \leqslant n : d(x_m, a) \leqslant e/2^k\}| > 1 - 1/2^k$ for each $n > n_k$. We can assume that $n_k < n_{k+1}$ for each $k \in \mathbb{N}$. A sequence $\{y_m\}$ is defined by

$$y_m = \begin{cases} x_m, & 1 \leqslant m \leqslant n_1, \\ x_m, & n_k < m \leqslant n_{k+1}, d(x_m, a) \leqslant e/2^k, \\ a, & \text{otherwise.} \end{cases}$$

Let $c \in E^+$. Choose $k \in \mathbb{N}$ with $e/2^k \ll c$. Then $d(y_m, a) \ll c$ for each $m > n_k$. Hence $\lim_{m\to\infty} y_m = a$.

For each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $1/2^k < \varepsilon$. Fix $n \in \mathbb{N}$. If $n_k < n \leqslant n_{k+1}$, then

$$\{m \leqslant n : y_m \neq x_m\} \subset \{1, 2, \cdots, n\} - \{m \leqslant n : d(x_m, a) \leqslant \frac{e}{2^k}\},$$

thus

$$\frac{1}{n}|\{m \leqslant n : y_m \neq x_m\}| \leqslant 1 - \frac{1}{n}|\{m \leqslant n : d(x_m, a) \leqslant \frac{e}{2^k}\}| < \frac{1}{2^k} < \varepsilon.$$

Hence $\lim_{n\to\infty} \frac{1}{n}|\{m \leqslant n : y_m \neq x_m\}| = 0$, i.e., $\delta\{m \in \mathbb{N} : y_m \neq x_m\} = 0$, therefore $x_m = y_m$ for almost all $m \in \mathbb{N}$.

(2) ⇒ (3). Suppose that $\{y_n\}$ is a convergent sequence in X such that $x_n = y_n$ for almost all $n \in \mathbb{N}$. Let $A = \{n \in \mathbb{N} : x_n = y_n\}$. Then $\delta(A) = 1$. Thus $\{y_n\}_{n\in A}$ is both a convergent sequence and a statistically dense subsequence of $\{x_n\}$.

(3) ⇒ (4) is obvious. Next, we prove that (4) ⇒ (1). Suppose there is a statistically dense subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that the sequence $\{x_{n_k}\}$ is statistically convergent. Set $st\text{-}\lim_{k\to\infty} x_{n_k} = a \in X$ and let $A = \{n_k : k \in \mathbb{N}\}$. Then $\delta(A) = 1$. Since, for each $c \in E^+$,

$$\{n \in \mathbb{N} : d(x_n, a) \leqslant c\} \supset \{n_k \in \mathbb{N} : d(x_{n_k}, a) \leqslant c\},$$

and

$$\delta(\{n \in \mathbb{N} : d(x_n, a) \leqslant c\}) \geqslant \delta(\{n_k \in \mathbb{N} : d(x_{n_k}, a) \leqslant c\}) = 1,$$

we have $st\text{-}\lim_{n\to\infty} x_n = a$. □

Corollary 2.9. *Every statistically convergent sequence has a convergent subsequence in a cone metric space.*

The converse of Corollary 2.9 is not hold, i.e., there is a non-statistically convergent sequence in a cone metric space such that it has a convergent subsequence. In fact, the sequence $\{x_n\}$ in Example 2.4 has the convergent subsequence $\{x_{m^3}\}$, but the $\{x_n\}$ is not statistically convergent.

Corollary 2.10. *Every statistically complete cone metric space is complete.*

Proof. Let (X, d) be a statistically complete cone metric space. If a sequence $\{x_n\}$ is a Cauchy sequence in (X, d), then $\{x_n\}$ is a statistical Cauchy sequence in X. Since X is statistically complete, the sequence $\{x_n\}$ is statistically convergent. There is a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $\{x_{n_k}\}$ converges to a point $x \in X$ by Corollary 2.9.

For each $c \in E^+$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c/2$ for each $n, m > n_0$. Since $\lim_{k\to\infty} x_{n_k} = x$, there exists $k_0 \in \mathbb{N}$ such that $n_{k_0} > n_0$ and $d(x_{n_{k_0}}, x) \ll c/2$. Then $d(x_n, x) \leqslant d(x_n, x_{n_{k_0}}) + d(x_{n_{k_0}}, x) \ll c$ for each $n > n_0$, i.e., $\lim_{n\to\infty} x_n = x$. Hence, (X, d) is a complete cone metric space. □

Definition 2.11. A sequence $\{x_n\}$ in a cone metric space (X, d) is said to be *statistically bounded* if there exist $a \in X$ and $c \in E^+$ such that $\delta\{n \in \mathbb{N} : d(x_n, a) \leqslant c\} = 1$.

Theorem 2.12. *Every statistical Cauchy sequence is statistically bounded in a cone metric space.*

Proof. Let $\{x_n\}$ be a statistical Cauchy sequence in a cone metric space (X, d). Fix $a \in X$, and $c \in E^+$. There exists $n_0 \in \mathbb{N}$ such that $\delta\{n \in \mathbb{N} : d(x_n, x_{n_0}) \leqslant c\} = 1$.

Since $c \in \mathrm{int}P$ and $d(x_{n_0}, a) \in P$, there is an open neighborhood U of c in E such that $U \subset P$, thus $U + d(x_{n_0}, a) \subset P$ by Definition 1.1(C2). Note that $U + d(x_{n_0}, a)$ is an open subset of E, and $c + d(x_{n_0}, a) \in U + d(x_{n_0}, a) \subset P$. So $c + d(x_{n_0}, a) \in \mathrm{int}P$, i.e., $c + d(x_{n_0}, a) \gg 0$. Put $e = c + d(x_{n_0}, a)$. Then $e \in E^+$.

If $d(x_k, x_{n_0}) \leqslant c$, then

$$d(x_k, a) \leqslant d(x_k, x_{n_0}) + d(x_{n_0}, a) \leqslant c + d(x_{n_0}, a) = e,$$

thus for each $n \in \mathbb{N}$,

$$\{k \leqslant n : d(x_k, a) \leqslant e\} \supset \{k \leqslant n : d(x_k, x_{n_0}) \leqslant c\},$$

and

$$\delta\{n \in \mathbb{N} : d(x_n, a) \leqslant e\} \geqslant \delta\{n \in \mathbb{N} : d(x_n, x_{n_0}) \leqslant c\} = 1.$$

Consequently, the sequence $\{x_n\}$ is statistically bounded in X. □

Corollary 2.13. *Every statistically convergent sequence is statistically bounded in a cone metric space.*

By Example 2.3 we already know that a statistically convergent sequence need not to be convergent. We can prove that every slowly oscillating and statistically convergent sequence is convergent in a cone metric space by a similar method in [4].

3. Statistically complete cone metric spaces

We give some definitions related to statistical completeness of cone metric spaces in accordance with a method in [4].

Definition 3.1. Let (X, d) be a cone metric space, and $F \subset X$. Put

$$\mathrm{cl}_{st}F = \{x \in X : \text{there is a sequence } \{x_n\} \text{ in } F \text{ such that } st\text{-}\lim_{n\to\infty} x_n = x\},$$

$$F^{st\text{-d}} = \mathrm{cl}_{st}(F \setminus \{x\}).$$

The sets $\mathrm{cl}_{st}F$, $F^{st\text{-d}}$ are called the *statistically-sequential closure* of F in X and the *statistically-sequential derived set* of F in X, respectively. A set F is said to be *statistically-sequentially closed* if $F = \mathrm{cl}_{st}F$. Every point in $F^{st\text{-d}}$ is called a *statistically-sequential accumulation point* of F.

A point $x \in X$ is called a *statistically-sequential accumulation point* of a sequence $\{x_n\}$ in a cone metric space (X, d), if there is a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $st\text{-}\lim_{k\to\infty} x_{n_k} = x$.

Definition 3.2. Let (X, d) be a cone metric space.

(1) A subset F of X is said to be *statistically-sequentially countably compact* if any infinite subset of F has at least one statistically-sequentially accumulation point in F.
(2) A subset F of X is said to be *statistically-sequentially compact* if whenever $\{x_n\}$ is a sequence in F there is a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $st\text{-}\lim_{k\to\infty} x_{n_k} = x \in F$.

Definition 3.3. ([20]) Let $\mathcal{A}$ be an open cover of a cone metric space (X, d). An element $c \in E^+$ is called a *Lebesgue element* for the cover $\mathcal{A}$ if a subset B of X has an upper bound c, then $B \subset A$ for some $A \in \mathcal{A}$.

Lemma 3.4. *Every open cover of a statistically-sequentially compact cone metric space has a Lebesgue element.*

Proof. Let $\mathcal{A} = \{A_\alpha\}_{\alpha\in I}$ be an open cover of a statistically-sequentially compact cone metric space (X, d). We can assume that $X \notin \mathcal{A}$. Suppose that $\mathcal{A}$ dose not have a Lebesgue element. Fix $c \in E^+$. Then, for each $n \in \mathbb{N}$, there is a non-empty subset B_n of X such that $\frac{c}{n}$ is an upper bound of B_n and $B_n \not\subset A_\alpha$ for each $\alpha \in I$. Take $x_n \in B_n$ for each $n \in \mathbb{N}$. Since X is statistically-sequentially compact, there is a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $st\text{-}\lim_{k\to\infty} x_{n_k} = p \in X$, and there exists a subsequence $\{x_{n_{k_m}}\}$ of the sequence $\{x_{n_k}\}$ such that $\lim_{m\to\infty} x_{n_{k_m}} = p \in X$ by Corollary 2.9. Since $\mathcal{A}$ is an open cover of X, then $p \in A_{\alpha_0}$ for some $\alpha_0 \in I$. Find $c_1 \in E^+$ such that $B(p, c_1) \subset A_{\alpha_0}$. Then there is $m_0 \in \mathbb{N}$ such that $d(p, x_{n_{k_{m_0}}}) \leqslant \frac{c_1}{2}$ and $\frac{2c}{n_{k_{m_0}}} \ll c_1$.

If $x \in B_{n_{k_{m_0}}}$, then $d(x, p) \leqslant d(x, x_{n_{k_{m_0}}}) + d(x_{n_{k_{m_0}}}, p) \leqslant \frac{c}{n_{k_{m_0}}} + \frac{c_1}{2} \ll \frac{c_1}{2} + \frac{c_1}{2} = c_1$, i.e., $B_{n_{k_{m_0}}} \subset B(p, c_1) \subset A_{\alpha_0}$, a contradiction. Hence, $\mathcal{A}$ has a Lebesgue element. □

Theorem 3.5. *The following are equivalent for a cone metric space (X, d):*

(1) *X is statistically-sequentially compact;*
(2) *X is statistically-sequentially countably compact;*
(3) *X is compact;*
(4) *X is countably compact.*

Proof. (1) ⇒ (3). Let (X, d) be a statistically-sequentially compact cone metric space. First, we show that there exists a finite covering of X consisting of open c-balls $\{B(x, c)\}_{x\in X}$ for each $c \in E^+$. If not, there exists $c \in E^+$ such that X cannot be covered by finitely many c-balls. Construct a sequence $\{x_n\}$ in X as follows: First, fix a point $x_1 \in X$, and take a point $x_2 \in X \setminus B(x_1, c)$ by $X \neq B(x_1, c)$. In general, given $\{x_i\}_{i\leqslant n}$ in X, choose a point $x_{n+1} \in X \setminus \bigcup_{i\leqslant n} B(x_i, c)$ because $X \neq \bigcup_{i\leqslant n} B(x_i, c)$. Then $d(x_{n+1}, x_i) \not\ll c$ for each $i \leqslant n$, thus $\{x_n : n \in \mathbb{N}\}$ is a closed discrete subspace of X.

Secondly, we prove that X is compact. Let $\mathcal{U}$ be an open cover of X. There is $\delta \in E^+$ such that δ is a Lebesgue element for the open cover $\mathcal{U}$ by Lemma 3.4. Put $c = \frac{\delta}{3}$. There exists a finite subset F of X such that $X = \bigcup_{x\in F} B(x, c)$. For each $x \in F$, since $2c$ is an upper bound of the set $B(x, c)$, there is $U_x \in \mathcal{U}$ such that $B(x, c) \subset U_x$. Therefore, $\{U_x\}_{x\in F}$ is a finite subcover of $\mathcal{U}$. Hence, X is compact.

(3) $\Rightarrow$ (4) is obvious. (4) $\Rightarrow$ (2) is hold since every cone metric space is first-countable.

(2) $\Rightarrow$ (1). Suppose that X is statistically-sequentially countably compact. Let $\{x_n\}$ be a sequence in X. Put $A = \{x_n : n \in \mathbb{N}\}$, and assume that A is an infinite set. Since X is statistically-sequentially countably compact, there is $x \in A^{st\text{-d}}$. Then there is a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $st\text{-}\lim_{k\to\infty} x_{n_k} = x$. This completes the proof. □

Theorem 3.6. *Let (X, d) be a cone metric space, and $F \subset X$.*

(1) *If F is statistically-sequentially compact, then F is statistically-sequentially closed.*
(2) *If X is statistically-sequentially compact and F is statistically-sequentially closed, then F is statistically-sequentially compact.*

Proof. (1) Suppose that F is a statistically-sequentially compact subset of X. Take any $x \in \mathrm{cl}_{st}F$. There is a sequence $\{x_n\}$ in F such that $st\text{-}\lim_{n\to\infty} x_n = x$. Since F is statistically-sequentially compact, there is a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $st\text{-}\lim_{k\to\infty} x_{n_k} = x' \in F$. Then $x = x' \in F$ by (1) of Lemma 2.5. Thus F is statistically-sequentially closed.

(2) Suppose that F is a statistically-sequentially closed subset of a statistically-sequentially compact cone metric space X. If $\{x_n\}$ is a sequence in F, since X is statistically-sequentially compact, there is a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $st\text{-}\lim_{k\to\infty} x_{n_k} = x \in X$. Since F is statistically-sequentially closed, $x \in F$, i.e., $st\text{-}\lim_{k\to\infty} x_{n_k} = x \in F$. Hence F is statistically-sequentially compact in X. □

Let (X, d) be a cone metric space. A subset A of X is said to be *upper bounded* [20] if there exists $c \in E^+$ such that $d(x, y) \leqslant c$ for all $x, y \in A$; the c is called an *upper bound* of A.

A useful criterion for statistical completeness of cone metric spaces is the following.

Theorem 3.7. *A cone metric space (X, d) is statistically complete if and only if for every decreasing sequence $\{F_n\}$ of the statistically-sequentially closed non-empty subsets of X, if there is a sequence $\{b_n\}$ converging to 0 in E such that b_n is an upper bound of the set F_n for each $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} F_n$ is a single-point set.*

Proof. Assume the cone metric space (X, d) is statistically complete. Let $\{F_n\}$ be a decreasing sequence of the statistically-sequentially closed non-empty subsets of X such that there is a sequence $\{b_n\}$ converging to 0 in E such that b_n is an upper bound of the set F_n for each $n \in \mathbb{N}$. Choose a sequence $\{x_n\}$ in X such that $x_n \in F_n, \forall n \in \mathbb{N}$. Then the sequence $\{x_n\}$ is a statistical Cauchy sequence. In fact, for each $c \in E^+$, since $\lim_{n\to\infty} b_n = 0$, there is $n_0 \in \mathbb{N}$ such that $b_n \ll c$ for all $n \geqslant n_0$. Thus $x_n, x_m \in F_{n_0}$ for each $n, m \geqslant n_0$, therefore $d(x_n, x_m) \leqslant b_{n_0} \ll c$. Consequently, it follows that $\{x_n\}$ is a Cauchy sequence, so a statistical Cauchy sequence.

By the statistical completeness of (X, d), the sequence $\{x_n\}$ is statistically convergent, say $st\text{-}\lim_{n\to\infty} x_n = x$. For each $n \in \mathbb{N}$, since $\{x_k : k \geqslant n\} \subset F_n$ and F_n is statistically-sequentially closed in X, we have $x \in F_n$. Thus $x \in \bigcap_{n\in\mathbb{N}} F_n$. If $y \in \bigcap_{n\in\mathbb{N}} F_n$, then $x, y \in F_n$ for each $n \in \mathbb{N}$. Thus $d(x, y) \ll c$ for each $c \in E^+$. Hence, we have $d(x, y) = 0$, therefore $x = y$, i.e., $\bigcap_{n\in\mathbb{N}} F_n$ is a single-point set.

Conversely, suppose that $\{x_n\}$ is a statistical Cauchy sequence in (X, d). Fix $e \in E^+$. For each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $\delta\{n \in \mathbb{N} : d(x_n, x_{n_k}) \leqslant \frac{e}{2^{k+4}}\} = 1$. We can assume $n_k < n_{k+1}$ and $d(x_{n_{k+1}}, x_{n_k}) \leqslant \frac{e}{2^{k+4}}$ for each $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let $b_k = \frac{e}{2^k}$ and

$$F_k = \mathrm{cl}_{st}B(x_{n_k}, \frac{e}{2^{k+2}}) = \mathrm{cl}_{st}\{y \in X : d(x_{n_k}, y) \ll \frac{e}{2^{k+2}}\}.$$

Then $\lim\limits_{k\to\infty} b_k = 0$ and b_k is an upper bound of F_k. If $y \in F_{k+1}$, then $d(y, x_{n_{k+1}}) \leqslant \frac{e}{2^{k+3}}$ and $d(x_{n_{k+1}}, x_{n_k}) \leqslant \frac{e}{2^{k+4}}$, thus

$$d(y, x_{n_k}) \leqslant d(y, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k}) \leqslant \frac{e}{2^{k+2}}.$$

Therefore $y \in F_k$, i.e., $F_{k+1} \subset F_k$. Thus there exists $x \in \bigcap_{n\in\mathbb{N}} F_n$.

We prove that the sequence $\{x_n\}$ statistically converges to x. For any $c \in E^+$, since $\lim\limits_{n\to\infty} \frac{e}{2^n} = 0$ in E, there exists $k \in \mathbb{N}$ such that $\frac{e}{2^n} \ll c$ for each $n \geqslant k$. We have $d(x, x_{n_k}) \leqslant \frac{e}{2^{k+2}}$ by $x \in F_k$. Thus for each $n > n_k$, if $d(x_n, x_{n_k}) \leqslant \frac{e}{2^{k+4}}$, then

$$d(x_n, x) \leqslant d(x_n, x_{n_k}) + d(x_{n_k}, x) \leqslant \frac{e}{2^{k+4}} + \frac{e}{2^{k+2}} \leqslant \frac{e}{2^k} \ll c.$$

Therefore for each $m \in \mathbb{N}$,

$$\{n \leqslant m : d(x_n, x) \leqslant c\} \supset \{n \leqslant m : d(x_n, x_{n_k}) \leqslant \frac{e}{2^{k+4}}\},$$

and

$$\delta\{n \in \mathbb{N} : d(x_n, x) \leqslant c\} \geqslant \delta\{n \in \mathbb{N} : d(x_n, x_{n_k}) \leqslant \frac{e}{2^{k+4}}\} = 1.$$

Thus $\delta\{n \in \mathbb{N} : d(x_n, x) \leqslant c\} = 1$, so the sequence $\{x_n\}$ statistically converges to x. Hence (X, d) is statistically complete. □

Corollary 3.8. *Every compact cone metric space is statistically complete.*

Proof. Let (X, d) be a compact cone metric space. Suppose that $\{F_n\}$ is a decreasing sequence of statistically-sequentially closed non-empty subsets of X, and there is a sequence $\{b_n\}$ converging to 0 in E such that b_n is an upper bound of the set F_n for each $n \in \mathbb{N}$. Since each F_n is statistically-sequentially closed, it follows from Theorem 3.6 that F_n is compact in X, thus $\bigcap_{n\in\mathbb{N}} F_n \neq \varnothing$. Since b_n is an upper bound of the set F_n for each $n \in \mathbb{N}$, and the sequence $\{b_n\}$ converges to 0 in E, it is easy to see that $\bigcap_{n\in\mathbb{N}} F_n$ is a single-point set. Hence, (X, d) is statistically complete by Theorem 3.7. □

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