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Sequentially compact spaces with a point-countable k-network [☆]

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ABSTRACT

Countably compact spaces with a point-countable k-network are not necessarily metrizable. Is every sequentially compact space with a point-countable k-network metrizable? This paper gives an affirmative answer to the question.

1. Introduction

Since A. Miščenko [14] proved that every countably compact space with a point-countable base is metrizable, point-countable covers have played an important role in the development of general topology [10]. Much research in some branches, such as generalized metric spaces [8], function spaces [12], and topological algebra [16] etc., concerned the topological property of being a space with a point-countable k-network.

Metrization is a fundamental and considerably important research subject in general topology. Recently, some nice metrization theorems have been obtained in special topological spaces, for example in mono-

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tonically metacompact spaces [6], *JMP*-spaces [3], and spaces with weakly uniform weak bases [18]. It is well known that every compact space with a point-countable k-network is metrizable [9], but there exists a countably compact space with a point-countable k-network which is not metrizable [9]. Is every sequentially compact space with a point-countable k-network metrizable? In this paper, we give an affirmative answer to the question by proving a much wider result, i.e., every totally countably compact space with a point-countable p–k-network is compact metrizable.

First of all, let's recall some concepts.

Assume $\mathcal{P}$ is a family of subsets of a topological space X. Let's denote

$$\mathcal{P}^{<\omega} = \{\mathcal{F} \subset \mathcal{P} : |\mathcal{F}| < \omega\} \text{ and}$$
$$(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\} \text{ for every } x \in X.$$

Definition 1.1. Let $\mathcal{P}$ be a family of subsets of a topological space X.

(1) $\mathcal{P}$ is a *network* [2] for X if for every point $x \in X$ and any neighbourhood U of x, there exists a $P \in \mathcal{P}$ such that $x \in P \subset U$.
(2) $\mathcal{P}$ is a *k-network* [15] for X if whenever K is a compact subset of an open set U in X, there exists an $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $K \subset \cup\mathcal{F} \subset U$.
(3) $\mathcal{P}$ is a *p-meta-base* [5,11] for X if for any two distinct points $x, y \in X$, there exists an $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $x \in (\cup\mathcal{F})^\circ \subset \cup\mathcal{F} \subset X \setminus \{y\}$.
(4) $\mathcal{P}$ is a *p–k-network* [9,11] for X if whenever K is compact and $y \in X \setminus K$, there exists an $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $K \subset \cup\mathcal{F} \subset X \setminus \{y\}$.

Remark 1.2.

(1) The senses of "p-meta-base" and "p–k-network" originally emerged in [5] and [9], respectively. For the sake of brevity, the second author in this paper inspired by the concept of "p-base" in [4] first used the terminology "p-meta-base" and "p–k-network" in his monograph [11].
(2) It is easy to see that every base of a space is a p-meta-base and k-network, every p-meta-base or k-network of a space is a p–k-network, and every k-network of a space is a network.

Definition 1.3. A topological space X is called *totally countably compact* [17] if every sequence of X contains a subsequence with the compact closure.

Remark 1.4. A space is *ω-bounded* if every countable subset has compact closure. Every compact space is ω-bounded, and every ω-bounded space is totally countably compact. Every sequentially compact space is totally countably compact, and every totally countably compact space is countably compact. Totally countable compactness has drawn topologists' attention in the theory of paratopological groups [1].

In what follows, all topological spaces are assumed to be Hausdorff in the paper. For some terminology unstated here, readers may refer to [7].

2. Main results

Let A be a subset of a set X and $\mathcal{P}$ be a cover of the set A. The family $\mathcal{P}$ is said to be a *minimal cover* of A if $\mathcal{P} \setminus \{P\}$ is not a cover of A for every $P \in \mathcal{P}$.

Lemma 2.1. *([14]) Suppose that $\mathcal{P}$ is a point-countable cover of a set X. Then every non-empty subset of X has at most countably many minimal finite covers consisting of elements of $\mathcal{P}$.*

Theorem 2.2. *If X is a totally countably compact space with a point-countable p–k-network, then X is a compact metrizable space.*

Proof. Let $\mathcal{P}$ be a point-countable p–k-network for X. We can assume that $\mathcal{P}$ is closed under finite intersections.

Claim 1. *$\mathcal{P}$ is a p-meta-base for X.*

Let x, y be two distinct points of X. The space X being Hausdorff, we can choose a closed neighbourhood Z of x in X such that $Z \subset X \setminus \{y\}$. Put

$$\mathcal{Q} = \{P \in \mathcal{P} : y \notin P\} \text{ and}$$
$$(\mathcal{Q})_z = \{P_n(z) : n \in \mathbb{N}\} \text{ for every } z \in Z.$$

We shall show that there exists an $\mathcal{F} \in \mathcal{Q}^{<\omega}$ such that $Z \subset \cup\mathcal{F}$. In fact, if no finite subfamily of $\mathcal{Q}$ covers the set Z, then there exists a sequence $\{z_n\}$ of points of Z such that $z_n \notin P_j(z_i)$ for every $n \in \mathbb{N}$ and every $i, j < n$. The space X being totally countably compact, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $B = \overline{\{z_{n_k} : k \in \mathbb{N}\}}$ is a compact subset of X. Since $B \subset X \setminus \{y\}$ and $\mathcal{P}$ is a p–k-network for X, there exists a $\mathcal{Q}' \in \mathcal{Q}^{<\omega}$ such that $B \subset \cup\mathcal{Q}'$. Obviously, there exist a $Q \in \mathcal{Q}'$ and a subsequence $\{z_{n_{k_m}}\}$ of $\{z_{n_k}\}$ such that $\{z_{n_{k_m}} : m \in \mathbb{N}\} \subset Q$. Because of $z_{n_{k_1}} \in Q$, we may choose a $j \in \mathbb{N}$ such that $Q = P_j(z_{n_{k_1}})$. Hence, $z_n \notin P_j(z_{n_{k_1}}) = Q$ for every $n > \max\{n_{k_1}, j\}$. This contradicts the fact that $z_{n_{k_m}} \in Q$ for every $m \in \mathbb{N}$. Thus

$$x \in Z^\circ \subset (\cup\mathcal{F})^\circ \subset \cup\mathcal{F} \subset X \setminus \{y\}.$$

Namely, $\mathcal{P}$ is a p-meta-base for X.

Further, the family $\mathcal{P}$ has following property (∗).

$$\text{If } x \in X \setminus A \text{ and } |A| < \omega, \text{ then there exists an } \mathcal{F} \in \mathcal{P}^{<\omega}$$
$$\text{such that } x \in (\cup\mathcal{F})^\circ \subset \cup\mathcal{F} \subset X \setminus A. \tag{$*$}$$

Put

$$\mathcal{H} = \cup\{\mathcal{F} \in \mathcal{P}^{<\omega} : \mathcal{F} \text{ is a minimal cover of } X\}.$$

Then $\mathcal{H}$ is countable by Lemma 2.1.

Claim 2. *$\mathcal{H}$ is also a p-meta-base for X.*

Let x, y be two distinct points of X. By Claim 1, there exists an $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $x \in (\cup\mathcal{F})^\circ \subset \cup\mathcal{F} \subset X \setminus \{y\}$. We can assume $x \notin [\cup(\mathcal{F} \setminus \{F\})]^\circ$, whence $(\cup\mathcal{F})^\circ \not\subset \cup(\mathcal{F} \setminus \{F\})$ for every $F \in \mathcal{F}$. In order to prove that $\mathcal{H}$ is a p-meta-base for X, it suffices to show $\mathcal{F} \in \mathcal{H}^{<\omega}$. For every $F \in \mathcal{F}$, pick

$$x(F) \in (\cup\mathcal{F})^\circ \setminus \cup(\mathcal{F} \setminus \{F\}) = F \cap (\cup\mathcal{F})^\circ \setminus \cup(\mathcal{F} \setminus \{F\}).$$

Put $A = \{x(F) : F \in \mathcal{F}\}$. Then $|A| < \omega$. For every $z \in X \setminus A$, there exists an $\mathcal{F}_z \in \mathcal{P}^{<\omega}$ such that $z \in (\cup\mathcal{F}_z)^\circ \subset \cup\mathcal{F}_z \subset X \setminus A$ by property (∗). Let

$$\mathcal{R} = \mathcal{F} \cup \bigcup_{z \in X \setminus A} \mathcal{F}_z.$$

We shall show that there exists a $\mathcal{B} \in \mathcal{R}^{<\omega}$ such that $X = \cup\mathcal{B}$. Suppose no finite subfamily of $\mathcal{R}$ covers the space X. For every $p \in X$, denote

$$(\mathcal{R})_p = \{P_i(p) : i \in \mathbb{N}\}.$$

Then there exists a sequence $\{x_n\}$ of points of X such that $x_n \notin P_i(x_j)$ for every $n \in \mathbb{N}$ and every $i, j < n$. The space X being countably compact, the sequence $\{x_n\}$ has a cluster point $a \in X$. Since

$$a \in X = A \cup (X \setminus A) = (\cup\mathcal{F})^\circ \cup \bigcup_{z \in X \setminus A} (\cup\mathcal{F}_z)^\circ,$$

there exists a $P \in \mathcal{R}$ such that P contains a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. This is a contradiction.

Without loss of generality, we can assume $\mathcal{B}$ is a minimal cover of X. Thus $\mathcal{B} \subset \mathcal{H}$. Further, $\mathcal{F} \subset \mathcal{B} \subset \mathcal{H}$ and $\mathcal{F} \in \mathcal{H}^{<\omega}$, since $(\mathcal{R})_{x(F)} = \{F\}$ for every $F \in \mathcal{F}$.

Claim 3. *X is a compact metrizable space.*

Since every countably compact space with a countable network is compact metrizable, it suffices to show that X has a countable network. Put

$$\mathcal{E} = \{X \setminus (\cup\mathcal{F})^\circ : \mathcal{F} \in \mathcal{H}^{<\omega}\} \text{ and}$$
$$\mathcal{C} = \{\cap\mathcal{E}' : \mathcal{E}' \in \mathcal{E}^{<\omega}\}.$$

Then both $\mathcal{E}$ and $\mathcal{C}$ are countable. We shall show $\mathcal{C}$ is a network for X. Let $x \in U$ with U open in X. By Claim 2, it follows that $\{x\} = \cap(\mathcal{E})_x$, whence

$$\{U\} \cup \{X \setminus E : E \in (\mathcal{E})_x\}$$

is a countable open cover of X. Since X is countably compact, there exists an $\mathcal{E}' \in \mathcal{E}^{<\omega}$ such that $x \in \cap\mathcal{E}' \subset U$. Hence, $\mathcal{C}$ is a network for X. □

Corollary 2.3. *Every sequentially compact space with a point-countable k-network is compact metrizable.*

Corollary 2.4. *([9]) Every countably compact k-space with a point-countable k-network is compact metrizable.*

Proof. Suppose that X is a countably compact k-space with a point-countable k-network. By Theorem 2.2, every compact subspace of X is metrizable. Since X is a k-space, X is a sequential space [13]. Then X is sequentially compact and so compact metrizable by Corollary 2.3. □

We conclude this paper with a question. Recall that a topological space X is of *countable tightness* if whenever $x \in \overline{A}$ in X, then $x \in \overline{C}$ for some countable $C \subset A$. Every sequential space is of countable tightness.

Question 2.5. *Suppose that X is a countably compact space with a point-countable k-network. If X is of countable tightness, is the space X compact metrizable?*

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