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CONTRACTIONS OF NADLER TYPE ON PARTIAL TVS-CONE METRIC SPACES

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Abstract. This paper introduces partial tvs-cone metric spaces as a common generalization of both tvs-cone metric spaces and partial metric spaces, and gives a new fixed point theorem for contractions of Nadler type on partial tvs-cone metric spaces. As corollaries, we obtain the main results of S. B. Nadler (1969), D. Wardowski (2011), S. Radenović et al. (2011) and H. Aydi et al. (2012) are deduced.

1. Introduction. Let (X,p) be a metric space and H be the Hausdorff metric on $CB(X)$, the collection of all nonempty, closed, bounded subsets of X. A set-valued mapping $T : X \to CB(X)$ is called a *contraction of Nadler type* if there is $k \in (0,1)$ such that $H(Tx,Ty) \le kp(x,y)$ for all $x, y \in X$; and it is said to have a *fixed point* if there is $x \in X$ such that $x \in Tx$.

The study of fixed points for contractions using the Hausdorff metric was initiated by S. B. Nadler [9] who proved the following theorem.

THEOREM 1.1 ([9]). *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a contraction of Nadler type. Then T has a fixed point.*

Recently, Nadler's theorem has been extended and generalized to cone metric spaces by D. Wardowski [12], to tvs-cone metric spaces by S. Radenović et al. [10] and to partial metric spaces by H. Aydi et al. [1]. Cone metric spaces, tvs-cone metric spaces and partial metric spaces were introduced by Huang–Zhang [4], Du [3] and Matthews [8], respectively.

DEFINITION 1.2 ([3, 4]). Let X be a non-empty set and (E,P) be an ordered Banach space (resp. ordered topological vector space) with zero vector θ. A mapping $d : X \times X \to P$ is called a *cone metric* (resp. *tvs-cone metric*) and (X,d) is called a *cone metric space* (resp. *tvs-cone metric space*) if the following are satisfied for all $x, y, z \in X$:

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(1) $d(x,y) = \theta \Leftrightarrow x = y$.
(2) $d(x,y) = d(y,x)$.
(3) $d(x,y) \le d(x,z) + d(z,y)$.

DEFINITION 1.3 ([8]). Let X be a non-empty set and $\mathbb{R}_{\ge 0}$ be the set of all nonnegative real numbers. A mapping $p : X \times X \to \mathbb{R}_{\ge 0}$ is called a *partial metric* and (X, p) is called a *partial metric space* if the following are satisfied for all $x, y, z \in X$:
(1) $x = y \Leftrightarrow p(x,x) = p(y,y) = p(x,y)$.
(2) $p(x,y) = p(y,x)$.
(3) $p(x,x) \le p(x,y)$.
(4) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$.

Note that tvs-cone metric spaces and partial metric spaces generalize metric spaces in two different directions. Furthermore, there is no inclusion between tvs-cone metric spaces and partial metric spaces. Thus, we are interested in extending and generalizing Theorem 1.1 to a common generalization of both tvs-cone metric spaces and partial metric spaces. In this paper, partial tvs-cone metric spaces are introduced naturally and the following implications hold:

$$\begin{array}{ccccc}
\text{metric} & \Rightarrow & \text{cone metric} & \Rightarrow & \text{tvs-cone metric} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{partial metric} & \Rightarrow & \text{partial cone metric} & \Rightarrow & \text{partial tvs-cone metric}
\end{array}$$

The following question is of interest.

QUESTION 1.4. *Can "metric" in Theorem* 1.1 *be relaxed to "partial tvs-cone metric"?*

In this paper, we answer the above question affirmatively. As corollaries of our result, we obtain the main results of S. B. Nadler [9], D. Wardowski [12], S. Radenović et. al. [10] and H. Aydi et. al. [1].

Throughout this paper, $\mathbb{N}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\ge 0}$ denote the set of all natural numbers, the set of all positive real numbers and the set of all nonnegative real numbers, respectively.

2. Ordered topological vector spaces

DEFINITION 2.1 ([3]). Let E be a topological vector space with zero vector θ. A subset P of E is called a *tvs-cone* in E if:

(1) P is non-empty and closed in E.
(2) $\alpha, \beta \in P$ and $a, b \in \mathbb{R}_{\ge 0} \Rightarrow a\alpha + b\beta \in P$.
(3) $\alpha, -\alpha \in P \Rightarrow \alpha = \theta$.

Definition 2.2 ([3]). Let P be a tvs-cone in a topological vector space E. The partial orderings $\leq$, $<$ and $\ll$ on E with respect to P are defined as follows, for $\alpha, \beta \in E$:

(1) $\alpha \leq \beta$ if $\beta - \alpha \in P$.
(2) $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.
(3) $\alpha \ll \beta$ if $\beta - \alpha \in P^\circ$, where P° denotes the interior of P in E.

The pair (E, P) is called an *ordered topological vector space.*

Remark 2.3 ([7]). Let (E, P) be an ordered topological vector space.

(1) It is known that $\theta \in P - P^\circ$, and we always suppose $P^\circ \neq \emptyset$.

(2) For convenience, we also use the notations "$\geq$", "$>$" and "$\gg$" in (E, P), with the obvious meaning. The following hold:

(a) $\alpha \geq \beta \Leftrightarrow \alpha - \beta \geq \theta \Leftrightarrow \alpha - \beta \in P$.
(b) $\alpha > \beta \Leftrightarrow \alpha - \beta > \theta \Leftrightarrow \alpha - \beta \in P - \{\theta\}$.
(c) $\alpha \gg \beta \Leftrightarrow \alpha - \beta \gg \theta \Leftrightarrow \alpha - \beta \in P^\circ$.
(d) $\alpha \gg \beta \Rightarrow \alpha > \beta \Rightarrow \alpha \geq \beta$.

Definition 2.4 ([11]). An ordered topological vector space (E, P) is called *strongly minihedral* [11] if every subset of E bounded from above has a supremum, or equivalently, every subset of E bounded from below has an infimum.

Remark 2.5. In this paper, all ordered topological vector spaces are assumed to be strongly minihedral. Let (E, P) be an ordered topological vector space and $F \subseteq E$.

(1) If F is bounded from above, then $\sup F$ exists and is finite.
(2) If F is not bounded from above, then $\sup F = +\infty$.
(3) If F is bounded from below, then $\inf F$ exists and is finite.
(d) If F is not bounded from below, then $\inf F = -\infty$.

Lemma 2.6 ([7]). *Let (E, P) be an ordered topological vector space.*

(1) *If $\alpha \gg \theta$, then $r\alpha \gg \theta$ for every $r \in \mathbb{R}_{\geq 0}$.*
(2) *If $\alpha \gg \theta$, then $\alpha \gg \frac{1}{2}\alpha \gg \cdots \gg \frac{1}{n}\alpha \gg \cdots \gg \theta$.*
(3) *If $\alpha_1 \gg \beta_1$ and $\alpha_2 \geq \beta_2$, then $\alpha_1 + \alpha_2 \gg \beta_1 + \beta_2$.*
(4) *If $\alpha \gg \beta \geq \gamma$ or $\alpha \geq \beta \gg \gamma$, then $\alpha \gg \gamma$.*
(5) *If $\alpha \gg \theta$ and $\beta \in E$, then there is $n \in \mathbb{N}$ such that $\frac{1}{n}\beta \ll \alpha$.*
(6) *If $\alpha \gg \theta$ and $\beta \gg \theta$, then there is $\gamma \gg \theta$ such that $\gamma \ll \alpha$ and $\gamma \ll \beta$.*

In order to investigate convergence for sequences in partial tvs-cone metric spaces, we need to introduce convergence for sequences in ordered topological vector spaces, which is different from the one in topological vector spaces.

Definition 2.7. Let (E, P) be an ordered topological vector space, $\{\alpha_n\}$ be a sequence in E and $\alpha \in E$. We say that $\{\alpha_n\}$ *converges to* α *in* (E, P) if for any $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $\alpha - \varepsilon \ll \alpha_n \ll \alpha + \varepsilon$ for all $n > n_0$. We then write $\widehat{\lim}_{n\to\infty} \alpha_n = \alpha$.

Lemma 2.8. *Let (E, P) be an ordered topological vector space, $\{\alpha_n\}$ be a sequence in E and $\alpha \in E$. If $\lim_{n\to\infty} \alpha_n = \alpha$, then $\widehat{\lim}_{n\to\infty} \alpha_n = \alpha$.*

Proof. Assume that $\lim_{n\to\infty} \alpha_n = \alpha$. Let $\varepsilon \gg \theta$, i.e., $\varepsilon \in P^\circ$. Then there is a neighborhood U of ε in E such that $U \subseteq P^\circ$. Set $U_1 = \alpha + \varepsilon - U$ and $U_2 = U + \alpha - \varepsilon$. Then U_1 and U_2 are neighborhoods of α in E. Since $\{\alpha_n\}$ converges to α, there is $n_0 \in \mathbb{N}$ such that $\alpha_n \in U_1 \cap U_2$ for all $n > n_0$. Let $n > n_0$.

(1) Since $\alpha_n \in U_1$, we have $\alpha_n = \alpha + \varepsilon - \beta_n$ for some $\beta_n \in U$. It follows that $\alpha + \varepsilon - \alpha_n = \beta_n \in U \subseteq P^\circ$. So $\alpha + \varepsilon - \alpha_n \gg \theta$, i.e., $\alpha_n \ll \alpha + \varepsilon$.

(2) Since $\alpha_n \in U_2$, we have $\alpha_n = \gamma_n + \alpha - \varepsilon$ for some $\gamma_n \in U$. It follows that $\alpha_n - \alpha + \varepsilon = \gamma_n \in U \subseteq P^\circ$. So $\alpha_n - \alpha + \varepsilon \gg \theta$, i.e., $\alpha_n \gg \alpha - \varepsilon$.

By (1) and (2), $\alpha - \varepsilon \ll \alpha_n \ll \alpha + \varepsilon$ for all $n > n_0$, so $\widehat{\lim}_{n\to\infty} \alpha_n = \alpha$. ∎

Remark 2.9. In [6, proof of Lemma 2.4], Z. Kadelburg, S. Radenović and V. Rakočević showed that the implication in Lemma 2.8 cannot be reversed even if (E, P) is an ordered Banach space.

Lemma 2.10. *Let (E, P) be an ordered topological vector space, and $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in E with $\widehat{\lim}_{n\to\infty} \alpha_n = \alpha$ and $\widehat{\lim}_{n\to\infty} \beta_n = \beta$. Then $\widehat{\lim}_{n\to\infty}(\alpha_n \pm \beta_n) = \alpha \pm \beta$.*

Proof. Let $\varepsilon \gg \theta$. Since $\widehat{\lim}_{n\to\infty} \alpha_n = \alpha$ and $\widehat{\lim}_{n\to\infty} \beta_n = \beta$, there is $n_0 \in \mathbb{N}$ such that $\alpha - \varepsilon/2 \ll \alpha_n \ll \alpha + \varepsilon/2$ and $\beta - \varepsilon/2 \ll \beta_n \ll \beta + \varepsilon/2$ for all $n > n_0$. It follows that $\alpha \pm \beta - \varepsilon \ll \alpha_n \pm \beta_n \ll \alpha \pm \beta + \varepsilon$ for all $n > n_0$. So $\widehat{\lim}_{n\to\infty}(\alpha_n \pm \beta_n) = \alpha \pm \beta$. ∎

Lemma 2.11. *Let (E, P) be an ordered topological vector space, and $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in E.*

(1) *Let $\alpha_n \geq \beta_n$ for all $n \in \mathbb{N}$. If $\widehat{\lim}_{n\to\infty} \alpha_n = \alpha$ and $\widehat{\lim}_{n\to\infty} \beta_n = \beta$, then $\alpha \geq \beta$.*
(2) *Let $\alpha_n \geq \beta_n \geq \gamma_n$ for all $n \in \mathbb{N}$. If $\widehat{\lim}_{n\to\infty} \alpha_n = \widehat{\lim}_{n\to\infty} \gamma_n = \alpha$, then $\widehat{\lim}_{n\to\infty} \beta_n = \alpha$.*

Proof. (1) For every $n \in \mathbb{N}$, set $\gamma_n = \alpha_n - \beta_n$. Then $\gamma_n \geq \theta$ and $\widehat{\lim}_{n\to\infty} \gamma_n = \alpha - \beta$ from Lemma 2.10. Set $\gamma = \alpha - \beta$. It suffices to prove that $\gamma \geq \theta$. First, we claim that if U is a neighborhood of θ, then there is $\varepsilon \gg \theta$ such that $\varepsilon \in U$. In fact, pick $\delta \gg \theta$. Then $\lim_{n\to\infty} \delta/n = \theta$, so there is $n_0 \in \mathbb{N}$ such that $\delta/n_0 \in U$. Set $\varepsilon = \delta/n_0$. Then $\varepsilon \gg \theta$ and $\varepsilon \in U$.

Now we prove that $\gamma \geq \theta$. If not, then $\gamma \notin P$, hence there is a neighborhood V of γ such that $V \cap P = \emptyset$ since P is closed. Note that $\widehat{\lim}_{n\to\infty} \gamma_n = \gamma$ and $\gamma_n \geq \theta$ for all $n \in \mathbb{N}$. For any $\varepsilon \gg \theta$, $\gamma + \varepsilon \gg \gamma_n \geq \theta$ for some $n \in \mathbb{N}$, hence $\gamma + \varepsilon \in P$. On the other hand, $V - \gamma$ is a neighborhood of θ. By the above claim, there is $\varepsilon_0 \gg \theta$ such that $\varepsilon_0 \in V - \gamma$. It follows that $\gamma + \varepsilon_0 \in V$, hence $\gamma + \varepsilon_0 \notin P$. This contradicts $\gamma + \varepsilon \in P$ for any $\varepsilon \gg \theta$.

(2) Let $\varepsilon \gg \theta$. Since $\widehat{\lim}_{n\to\infty} \alpha_n = \widehat{\lim}_{n\to\infty} \gamma_n = \alpha$, there is $n_0 \in \mathbb{N}$ such that $\alpha - \varepsilon \ll \alpha_n \ll \alpha + \varepsilon$ and $\alpha - \varepsilon \ll \gamma_n \ll \alpha + \varepsilon$ for all $n > n_0$. It follows that $\alpha - \varepsilon \ll \beta_n \ll \alpha + \varepsilon$ for all $n > n_0$. So $\widehat{\lim}_{n\to\infty} \beta_n = \alpha$. ∎

COROLLARY 2.12. *Let (E, P) be an ordered topological vector space. If $\theta \leq \alpha \leq \varepsilon$ for every $\varepsilon \gg \theta$, then $\alpha = \theta$.*

3. Partial tvs-cone metric spaces. In this section, we define partial tvs-cone metric spaces, a common generalization of both tvs-cone metric spaces and partial metric spaces.

DEFINITION 3.1. Let (E, P) be an ordered topological vector space and let X be a non-empty set. A mapping $p : X \times X \to P$ is called a *partial tvs-cone metric* and (X, p) is called a *partial tvs-cone metric space* if the following are satisfied for all $x, y, z \in X$:

(1) $x = y \Leftrightarrow p(x, x) = p(y, y) = p(x, y)$.
(2) $p(x, y) = p(y, x)$.
(3) $p(x, x) \leq p(x, y)$.
(4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Now we exhibit some partial tvs-cone metric spaces which are neither partial metric spaces nor tvs-cone metric spaces.

EXAMPLE 3.2. Let (E, P) be an ordered topological vector space.

(1) Set $X = P$ and define $p : X \times X \to P$ by

$$p(\alpha, \beta) = \sup\{\alpha, \beta\} \quad \text{for } \alpha, \beta \in X.$$

Then (X, p) is a partial tvs-cone metric space which is neither a partial metric space nor a tvs-cone metric space.

(2) Set $X = \{(\alpha, \beta) \in E \times E : \alpha \leq \beta\}$ and define $p : X \times X \to E$ by

$$p((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = \sup\{\beta_1, \beta_2\} - \inf\{\alpha_1, \alpha_2\}$$

for $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in X$. Then (X, p) is a partial tvs-cone metric space which is neither a partial metric space nor a tvs-cone metric space.

REMARK 3.3. Let (X, p) be a partial tvs-cone metric space.

(1) For $x, y \in X$, if $p(x, y) = \theta$, then $x = y$. In fact, $p(x, x) \leq p(x, y)$ by Definition 3.1(3), so $\theta \leq p(x, x) \leq \theta$. It follows that $p(x, x) = \theta$. Similarly,

$p(y,y)=\theta$. Consequently, $p(x,x)=p(y,y)=p(x,y)$. By Definition 3.1(1), $x=y$.

(2) However, $x=y\in X$ does not imply $p(x,y)=\theta$ by Example 3.2.

Proposition 3.4. *Let (X,p) be a partial tvs-cone metric space. Set $\mathscr{B}=\{B(x,\varepsilon):x\in X$ and $\varepsilon\gg\theta\}$, where $B(x,\varepsilon)=\{y\in X:p(x,y)\ll p(x,x)+\varepsilon\}$ for every $x\in X$ and every $\varepsilon\gg\theta$. Then $\mathscr{B}$ is a base for some topology on X.*

Proof. It is clear that $x\in B(x,\varepsilon)$ for every $x\in X$ and every $\varepsilon\gg\theta$, and so $X=\bigcup\mathscr{B}$. Let $z\in B(x,\alpha)\cap B(y,\beta)$, where $B(x,\alpha),B(y,\beta)\in\mathscr{B}$. Since $z\in B(x,\alpha)$, we have $p(x,z)\ll p(x,x)+\alpha$. Set $\gamma_1=p(x,x)+\alpha-p(x,z)$. Then $\gamma_1\gg\theta$.

We claim that $B(z,\gamma_1)\subseteq B(x,\alpha)$. In fact, if $u\in B(z,\gamma_1)$, then $p(z,u)\ll p(z,z)+\gamma_1$, hence $p(z,u)-p(z,z)\ll\gamma_1$. It follows that $p(x,u)\leq p(x,z)+p(z,u)-p(z,z)\ll p(x,z)+\gamma_1=p(x,x)+\alpha$, hence $u\in B(x,\alpha)$.

In the same way, we can show that there is $\gamma_2\gg\theta$ such that $B(z,\gamma_2)\subseteq B(y,\beta)$. Thus, there is $\gamma\gg\theta$ such that $\gamma\ll\gamma_1$ and $\gamma\ll\gamma_2$ from Lemma 2.6(6). Let $v\in B(z,\gamma)$. Then $p(z,v)\ll p(z,z)+\gamma\ll p(z,z)+\gamma_1$ and $p(z,v)\ll p(z,z)+\gamma\ll p(z,z)+\gamma_2$, so $v\in B(z,\gamma_1)\subseteq B(x,\alpha)$ and $v\in B(z,\gamma_2)\subseteq B(y,\beta)$, and hence $v\in B(x,\alpha)\cap B(y,\beta)$. This proves that $z\in B(z,\gamma)\subseteq B(x,\alpha)\cap B(y,\beta)$. Consequently, $\mathscr{B}$ is a base for some topology on X. In fact, $\mathscr{T}=\{U\subseteq X:$ there is $B'\subseteq\mathscr{B}$ such that $U=\bigcup\mathscr{B}'\}$ is a topology on X and $\mathscr{B}$ is a base for $\mathscr{T}$. ∎

In this paper, we always suppose that a partial tvs-cone metric space (X,p) is a topological space with the topology $\mathscr{T}$ described in Proposition 3.4. We show that every partial tvs-cone metric space is a T_0-space and give an example to show that a partial tvs-cone metric space need not be a T_1-space.

Proposition 3.5. *Let (X,p) be a partial tvs-cone metric space. Then (X,p) is a T_0-space.*

Proof. Let $x,y\in X$ and $x\neq y$. By Definition 3.1(3), $p(x,y)-p(x,x)\geq\theta$ and $p(x,y)-p(y,y)\geq\theta$. Further, $p(x,y)-p(x,x)>\theta$ or $p(x,y)-p(y,y)>\theta$ from Definition 3.1(1). Without loss of generality, we can assume that $p(x,y)-p(x,x)>\theta$. By Corollary 2.12, there is $\varepsilon\gg\theta$ such that "$p(x,y)-p(x,x)\leq\varepsilon$" does not hold, i.e., "$p(x,y)\leq p(x,x)+\varepsilon$" does not hold. Thus, "$p(x,y)\ll p(x,x)+\varepsilon$" does not hold from Remark 2.3(2). Consequently, $y\notin B(x,\varepsilon)$. This proves that (X,p) is a T_0-space. ∎

Example 3.6. Let (E,P) be an ordered topological vector space and let $X=\{x,y\}$. Pick $\alpha\in P^\circ$. Define $p:X\times X\to E$ as follows:

$$p(x,x)=p(x,y)=p(y,x)=\alpha\quad\text{and}\quad p(y,y)=\theta.$$

It is easy to check that (X,p) is a partial tvs-cone metric space. For any $\varepsilon \gg \theta$, $p(x,y) = \alpha \ll \alpha + \varepsilon = p(x,x) + \varepsilon$, so $y \in B(x,\varepsilon)$. This shows that (X,p) is not a T_1-space.

Now we give a relation between convergence of sequences in partial tvs-cone metric spaces and convergence of sequences in ordered topological vector spaces.

Definition 3.7. Let (X,p) be a partial tvs-cone metric space. A sequence $\{x_n\}$ in X is said to be *p-convergent* to $x \in X$ if $\{x_n\}$ converges to x in (X,p), which is denoted by $\text{p-}\lim_{n\to\infty} x_n = x$.

Proposition 3.8. *Let (X,p) be a partial tvs-cone metric space, $\{x_n\}$ a sequence in X, and $x \in X$. Then the following are equivalent:*

(1) $\text{p-}\lim_{n\to\infty} x_n = x$.
(2) $\widehat{\lim}_{n\to\infty} p(x,x_n) = p(x,x)$.

Proof. (1)⇒(2): Assume that $\text{p-}\lim_{n\to\infty} x_n = x$. Let $\varepsilon \gg \theta$. Then there is $n_0 \in \mathbb{N}$ such that for every $n > n_0$, $x_n \in B(x,\varepsilon)$, i.e., $p(x,x_n) \ll p(x,x)+\varepsilon$. Since $p(x,x) - \varepsilon \ll p(x,x) \le p(x,x_n)$, we have $p(x,x) - \varepsilon \ll p(x_n,x) \ll p(x,x) + \varepsilon$. So $\widehat{\lim}_{n\to\infty} p(x,x_n) = p(x,x)$.

(2)⇒(1): Assume that $\widehat{\lim}_{n\to\infty} p(x,x_n) = p(x,x)$. Let $\varepsilon \gg \theta$. Then there is $n_0 \in \mathbb{N}$ such that $p(x,x) - \varepsilon \ll p(x,x_n) \ll p(x,x) + \varepsilon$ for all $n > n_0$. It follows that $x_n \in B(x,\varepsilon)$ for all $n > n_0$. So $\text{p-}\lim_{n\to\infty} x_n = x$. ■

4. The main results. The following definition comes from [10, 12].

Definition 4.1. Let (X,p) be a partial tvs-cone metric space. $C(X)$ denotes the collection of all non-empty closed subsets of X. A mapping $H : C(X) \times C(X) \to P$ is called a *partial Hausdorff tvs-cone metric* on $C(X)$ if for any $A, B \in C(X)$ the following conditions are satisfied:

(1) If $H(A,B) = \theta$, then $A = B$.
(2) $H(A,B) = H(B,A)$.
(3) If $\varepsilon \gg \theta$ and $x \in A$, then $p(x,y) \le H(A,B) + \varepsilon$ for some $y \in B$.
(4) One of the following is satisfied:
   (i) If $\varepsilon \gg \theta$, then there is $x \in A$ such that $H(A,B) \le p(x,y) + \varepsilon$ for all $y \in B$.
   (ii) If $\varepsilon \gg \theta$, then there is $y \in B$ such that $H(A,B) \le p(x,y) + \varepsilon$ for all $x \in A$.

Let (X,p) be a partial tvs-cone metric space. For $A \subseteq X$ and $x \in X$, set $p(x,A) = \inf\{p(x,a) : a \in A\}$ and denote by $\overline{A}$ the closure of A in (X,p).

Lemma 4.2. *Let (X,p) be a partial tvs-cone metric space, $A \subseteq X$ and $a \in X$. Then $a \in \overline{A}$ if and only if $p(a,A) = p(a,a)$.*

Proof. Necessity: Let $a \in \overline{A}$. For any $\varepsilon \gg \theta$, $B(a,\varepsilon) \cap A \neq \emptyset$. Pick $b \in B(a,\varepsilon) \cap A$. Then $b \in A$ and $p(a,b) < p(a,a)+\varepsilon$. Thus, $p(a,A) = \inf\{p(a,x) : x \in A\} \leq p(a,b) \leq p(a,a)+\varepsilon$. By Lemma 2.11(1), $p(a,A) \leq p(a,a)$. On the other hand, for every $x \in A$, $p(a,x) \geq p(a,a)$, and so $p(a,A) = \inf\{p(a,x) : x \in A\} \geq p(a,a)$. Consequently, $p(a,A) = p(a,a)$.

Sufficiency: Let $p(a,A) = \inf\{p(a,x) : x \in A\} = p(a,a)$. Then for any $\varepsilon \gg \theta$, there is $b \in A$ such that $p(a,b) < p(a,a) + \varepsilon/2 \ll p(a,a) + \varepsilon$, hence $b \in B(a,\varepsilon)$, and so $B(a,\varepsilon) \cap A \neq \emptyset$. It follows that $a \in \overline{A}$. ∎

Lemma 4.3. *Let (X,p) be a partial tvs-cone metric space. For $A,B,C \in C(X)$, set $\delta(A,B) = \sup\{p(a,B) : a \in A\}$. Then:*

(1) $\delta(A,A) = \sup\{p(a,a) : a \in A\}$.
(2) $\delta(A,A) \leq \delta(A,B)$.
(3) $\delta(A,B) = \theta \Rightarrow A \subseteq B$.
(4) $\delta(A,B) \leq \delta(A,C) + \delta(C,B) - \inf\{p(c,c) : c \in C\}$.

Proof. (1) For every $a \in A$, $p(a,A) = \inf\{p(a,x) : x \in A\} = p(a,a)$. So $\delta(A,A) = \sup\{p(a,A) : a \in A\} = \sup\{p(a,a) : a \in A\}$.

(2) For every $a \in A$, $p(a,a) \leq p(a,b)$ for all $b \in B$, and so $p(a,a) \leq \inf\{p(a,b) : b \in B\} = p(a,B) \leq \delta(A,B)$. By (1), $\delta(A,A) = \sup\{p(a,a) : a \in A\} \leq \delta(A,B)$.

(3) Suppose that $\delta(A,B) = \theta$. Let $a \in A$. Then $p(a,B) = \theta$. Since $\theta \leq p(a,a) \leq p(a,B) = \theta$ from the proof of (2), we have $p(a,a) = \theta$, and so $p(a,B) = p(a,a)$. By Lemma 4.2, $a \in \overline{B}$. It follows that $a \in B$ since B is closed in X. This proves that $A \subseteq B$.

(4) If $a \in A$, $b \in B$ and $c \in C$, then $p(a,b) \leq p(a,c) + p(c,b) - p(c,c)$, hence $p(a,B) \leq p(a,c) + p(c,B) - p(c,c)$, and so $p(a,B) + p(c,c) \leq p(a,c) + p(c,B) \leq p(a,c) + \delta(C,B)$. Thus, for every $a \in A$,

$$\begin{aligned} p(a,B) + \inf\{p(c,c) : c \in C\} &\leq \inf\{p(a,c) : c \in C\} + \delta(C,B) \\ &= p(a,C) + \delta(C,B). \end{aligned}$$

It follows that $\delta(A,B) + \inf\{p(c,c) : c \in C\} \leq \delta(A,C) + \delta(C,B)$. ∎

Lemma 4.4. *Let (X,p) be a partial tvs-cone metric space. For $A,B,C \in C(X)$, set $H(A,B) = \sup\{\delta(A,B), \delta(B,A)\}$. Then:*

(1) $H(A,A) = \delta(A,A)$.
(2) $H(A,A) \leq H(A,B)$.
(3) $H(A,B) = H(B,A)$.
(4) $H(A,B) \leq H(A,C) + H(C,B) - \inf\{p(c,c) : c \in C\}$.
(5) $H(A,B) = \theta \Rightarrow A = B$.

Proof. (1) Obvious.

(2) By (1) and Lemma 4.3(2), $H(A,A) = \delta(A,A) \leq \delta(A,B) \leq H(A,B)$.

(3) Obvious.

(4) By Lemma 4.3(4),

$$\begin{aligned}
H(A,B) &= \sup\{\delta(A,B),\delta(B,A)\}\\
&\le \sup\big\{\delta(A,C)+\delta(C,B)-\inf\{p(c,c):c\in C\},\\
&\qquad\qquad \delta(B,C)+\delta(C,A)-\inf\{p(c,c):c\in C\}\big\}\\
&= \sup\{\delta(A,C)+\delta(C,B),\delta(B,C)+\delta(C,A)\}-\inf\{p(c,c):c\in C\}\\
&\le \sup\{\delta(A,C)+\delta(C,A)\}+\sup\{\delta(B,C)+\delta(C,B)\}-\inf\{p(c,c):c\in C\}\\
&= H(A,C)+H(C,B)-\inf\{p(c,c):c\in C\}.
\end{aligned}$$

(5) Let $H(A,B)=\theta$. Then $\delta(A,B)=\theta$ and $\delta(B,A)=\theta$. By Lemma 4.3(3), $A\subseteq B$ and $B\subseteq A$. It follows that $A=B$. ∎

LEMMA 4.5. *Let (X,p) be a partial tvs-cone metric space, $A,B\in\mathscr{F}(X)$ and $h>1$. Then, for every $a\in A$, there is $b\in B$ such that $p(a,b)\le hH(A,B)$.*

Proof. Let $a\in A$. Then $H(A,B)\ge\delta(A,B)=\sup\{p(x,B):x\in A\}\ge p(a,B)$.

If $H(A,B)=\theta$, then $A=B$ from Lemma 4.4(5). So $H(A,B)\ge p(a,B)$ $=p(a,A)=\inf\{p(a,b):b\in A\}=p(a,a)$. It follows that $p(a,b)\le hH(A,B)$ for $b=a\in A=B$.

If $H(A,B)>0$, then $(h-1)H(A,B)>0$. Since $p(a,B)=\inf\{p(a,y):y\in B\}$, there is $b\in B$ such that $p(a,B)+(h-1)H(A,B)\ge p(a,b)$. It follows that $p(a,b)\le p(a,B)+(h-1)H(A,B)\le H(A,B)+(h-1)H(A,B)=hH(A,B)$. ∎

We give the definitions of Cauchy sequences in partial tvs-cone metric spaces and complete partial tvs-cone metric spaces.

DEFINITION 4.6 ([2, 5]). Let (X,p) be a partial tvs-cone metric space and $\{x_n\}$ be a sequence in X.

(1) $\{x_n\}$ is called a *Cauchy sequence* in (X,p) if

$$\widehat{\lim_{n,m\to\infty}}\, p(x_n,x_m)=\alpha \quad \text{for some } \alpha\in E.$$

(2) (X,p) is said to be *complete* if for every Cauchy sequence $\{x_n\}$ there is $x\in X$ such that

$$p(x,x)=\widehat{\lim_{n\to\infty}}\, p(x,x_n)=\widehat{\lim_{n,m\to\infty}}\, p(x_n,x_m).$$

DEFINITION 4.7. Let (X,p) be a partial tvs-cone metric space and $T:X\to C(X)$ be a set-valued mapping. Then T is called a *contraction of Nadler type* if there exists $k\in(0,1)$ such that $H(Tx,Ty)\le kp(x,y)$ for all $x,y\in X$.

Definition 4.8. Let (X,p) be a partial tvs-cone metric space and $T : X \to C(X)$ be a set-valued mapping. Then $x \in X$ is called a *fixed point* for T if $x \in Tx$.

Now we give the main result of this paper.

Theorem 4.9. *Let (X,p) be a complete partial tvs-cone metric space, and let $T : X \to C(X)$ be a contraction of Nadler type. Then T has a fixed point $x \in X$ with $p(x,x) = \theta$.*

Proof. Let $x_0 \in X$ be arbitrary and fixed, and let $x_1 = Tx_0$. Since $1/\sqrt{k} > 1$, there is $x_2 \in Tx_1$ such that

$$p(x_1, x_2) \leq \frac{1}{\sqrt{k}} H(Tx_0, Tx_1)$$

from Lemma 4.5. Since $H(Tx_0, Tx_1) \leq kp(x_0, x_1)$,

$$p(x_1, x_2) \leq \frac{1}{\sqrt{k}} H(Tx_0, Tx_1) \leq \sqrt{k}\, p(x_0, x_1).$$

Similarly, for $x_2 \in Tx_1$, there is $x_3 \in Tx_2$ such that

$$p(x_2, x_3) \leq \frac{1}{\sqrt{k}} H(Tx_1, Tx_2) \leq \sqrt{k}\, p(x_1, x_2).$$

By induction, we obtain a sequence $\{x_n\}$ in X such that for every $n \in \mathbb{N}$, $x_n \in Tx_{n-1}$ and $p(x_n, x_{n+1}) \leq \sqrt{k}\, p(x_{n-1}, x_n)$, and hence $p(x_n, x_{n+1}) \leq (\sqrt{k})^n p(x_0, x_1)$.

Let $m \in \mathbb{N}$. Then

$$\begin{aligned}
\theta \leq p(x_n, x_{n+m}) \\
&\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_{n+m}) \\
&\leq (\sqrt{k})^n p(x_0, x_1) + (\sqrt{k})^{n+1} p(x_0, x_1) + \cdots + (\sqrt{k})^{n+m-1} p(x_0, x_1) \\
&= ((\sqrt{k})^n + (\sqrt{k})^{n+1} + \cdots + (\sqrt{k})^{n+m-1}) p(x_0, x_1) \\
&= \frac{(\sqrt{k})^n}{1 - \sqrt{k}} p(x_0, x_1).
\end{aligned}$$

Since $0 < k < 1$, we have $\lim_{n\to\infty} \frac{(\sqrt{k})^n}{1-\sqrt{k}} = 0$, and hence

$$\lim_{n\to\infty} \frac{(\sqrt{k})^n}{1 - \sqrt{k}} p(x_0, x_1) = \theta.$$

By Lemmas 2.12 and 2.8, $\widehat{\lim}_{n,m\to\infty} p(x_n, x_m) = \theta$. So $\{x_n\}$ is a Cauchy sequence in (X,p). Since (X,p) is complete,

$$p(x,x) = \widehat{\lim_{n\to\infty}}\, p(x, x_n) = \widehat{\lim_{n\to\infty}}\, p(x_n, x_n) = \theta.$$

Since $x_{n+1} \in Tx_n$,

$$\theta \leq p(x_{n+1}, Tx) \leq \delta(Tx_n, Tx) \leq H(Tx_n, Tx) \leq kp(x_n, x).$$

It follows that $\widehat{\lim}_{n\to\infty} p(x_{n+1}, Tx) = \theta$ from Lemma 2.11(2). It is easy to see that $p(x, Tx) \leq p(x, x_{n+1}) + p(x_{n+1}, Tx) - p(x_{n+1}, x_{n+1})$. By Lemmas 2.10 and 2.11(2), $p(x, Tx) = \theta$. It follows that $p(x, Tx) = \theta = p(x, x)$. By Lemma 4.2, $x \in \overline{Tx} = Tx$. ∎

By Theorem 4.9, and the diagram in Section 1, the following corollaries are obtained immediately, which are the main results of D. Wardowski [12], S. Radenović et al. [10] and H. Aydi et al. [1].

Corollary 4.10 ([12]). *Let (X, d) be a complete cone metric space and $T : X \to C(X)$ be a contraction of Nadler type. Then T has a fixed point.*

Corollary 4.11 ([10]). *Let (X, d) be a complete tvs-cone metric space and $T : X \to C(X)$ be a contraction of Nadler type. Then T has a fixed point.*

Corollary 4.12 ([1]). *Let (X, p) be a complete partial metric space and $T : X \to CB(X)$ be a contraction of Nadler type. Then T has a fixed point.*

The following example illustrates Theorem 4.9.

Example 4.13. Let (X, p) be the partial tvs-cone metric space described in Example 3.2(1). Define $T : X \to C(X)$ by $Tx = \{x/3\}$ for each $x \in X$.

(1) *(X, p) is complete.* Let $\{x_n\}$ be a Cauchy sequence in (X, p). Then $\widehat{\lim}_{n,m\to\infty} p(x_n, x_m) = x$ for some $x \in X$. It follows that

$$p(x, x) = \sup\{x, x\} = x = \widehat{\lim_{n,m\to\infty}} p(x_n, x_m).$$

To prove (X, p) is complete, it suffices to prove that $\widehat{\lim}_{n\to\infty} p(x, x_n) = p(x, x)$. Note that $\widehat{\lim}_{n\to\infty} x_n = \widehat{\lim}_{n\to\infty} \sup\{x_n, x_n\} = \widehat{\lim}_{n\to\infty} p(x_n, x_n) = x$. So, whenever $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $x - \varepsilon \ll x_n \ll x + \varepsilon$, hence $p(x, x_n) = \sup\{x, x_n\} \ll x + \varepsilon = p(x, x) + \varepsilon$, and then $x_n \in B(x, \varepsilon)$. This proves that p-$\lim_{n\to\infty} x_n = x$. By Proposition 3.8, $\widehat{\lim}_{n\to\infty} p(x, x_n) = p(x, x)$.

(2) *T is a contraction of Nadler type.* Let $x, y \in X$. Then

$$\begin{aligned} H(Tx, Ty) &= H(\{x/3\}, \{y/3\}) = p(x/3, y/3) = \sup\{x/3, y/3\} \\ &= \tfrac{1}{3} \sup\{x, y\} \leq \tfrac{1}{2} \sup\{x, y\} = \tfrac{1}{2} p(x, y). \end{aligned}$$

(3) *T has a fixed point $x \in X$ with $p(x, x) = \theta$.* In fact, $\theta \in X$ and $T\theta = \{\theta/3\} = \{\theta\}$. So $\theta \in T\theta$, and $p(\theta, \theta) = \sup\{\theta, \theta\} = \theta$.

5. Some open questions around convergent sequences. Let (X, p) be a partial tvs-cone metric space. In this section, we raise some questions around convergent sequences in (X, p) and (X, d), where (X, d) is a tvs-cone metric space described in the following proposition.

Proposition 5.1. *Let (X,p) be a partial tvs-cone metric space. For $x,y \in X$, set $d(x,y) = 2p(x,y) - p(x,x) - p(y,y)$. Then $d : X \times X \to E$ is a tvs-cone metric on X and (X,d) is a tvs-cone metric space.*

Proof. By Definition 1.2, items (1)–(4) below give the complete proof.

(1) For all $x,y \in X$, $p(x,x) \leq p(x,y)$ and $p(y,y) \leq p(x,y)$ from Definition 3.1(3), so $d(x,y) = 2p(x,y) - p(x,x) - p(y,y) = (p(x,y) - p(x,x)) + (p(x,y) - p(y,y)) \geq \theta$, i.e., $d(x,y) \in P$.

(2) For every $x \in X$, $d(x,x) = 2p(x,x) - p(x,x) - p(x,x) = \theta$. On the other hand, if $d(x,y) = \theta$ for $x,y \in X$, i.e., $(p(x,y) - p(x,x)) + (p(x,y) - p(y,y)) = \theta$. By Definition 3.1(3), $p(x,y) - p(x,x) \geq \theta$ and $p(x,y) - p(y,y) \geq \theta$, and therefore $p(x,y) - p(x,x) = \theta$ and $p(x,y) - p(y,y) = \theta$, i.e., $p(x,x) = p(y,y) = p(x,y)$. It follows that $x = y$ from Definition 3.1(1).

(3) For all $x,y \in X$, $d(x,y) = 2p(x,y) - p(x,x) - p(y,y) = 2p(y,x) - p(y,y) - p(x,x) = d(y,x)$.

(4) For all $x,y,z \in X$,

$$\begin{aligned} d(x,z) &= 2p(x,z) - p(x,x) - p(z,z) \\ &\leq 2(p(x,y) + p(y,z) - p(y,y)) - p(x,x) - p(z,z) \\ &= (2p(x,y) - p(x,x) - p(y,y)) + (2p(y,z) - p(y,y) - p(z,z)) \\ &= d(x,y) + d(y,z). \ \blacksquare \end{aligned}$$

Definition 5.2. Let (X,p) be a partial tvs-cone metric space and (X,d) be a tvs-cone metric on X described in Proposition 5.1.

(1) (X,d) is said to be *induced by* (X,p).
(2) A sequence $\{x_n\}$ in X is said to *converge to* $x \in X$ *in* (X,d) if for any $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $d(x,x_n) \ll \varepsilon$ for all $n > n_0$.

In the above sections, we use $\text{p-}\lim_{n\to\infty} x_n = x$ to denote that the sequence $\{x_n\}$ converges to x in (X,p). To avoid confusion, we use a different notation for convergent sequences in (X,d).

Definition 5.3. Let (X,d) be a tvs-cone metric space. A sequence $\{x_n\}$ in X is said to be *d-convergent* to $x \in X$ if $\{x_n\}$ converges to x in (X,d), which is denoted by $\text{d-}\lim_{n\to\infty} x_n = x$.

Proposition 5.4. *Let (X,p) be a partial tvs-cone metric space and (X,d) be the tvs-cone metric space induced by (X,p). Assume that $\{x_n\}$ is a sequence in X and $x \in X$. If* $\text{d-}\lim_{n\to\infty} x_n = x$, *then* $\text{p-}\lim_{n\to\infty} x_n = x$.

Proof. Let $\text{d-}\lim_{n\to\infty} x_n = x$. Whenever $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $d(x,x_n) \ll \varepsilon$ for all $n > n_0$. Let $n > n_0$. Then $2p(x,x_n) - p(x,x) - p(x_n,x_n) \ll \varepsilon$, i.e., $p(x,x_n) \leq p(x,x) + \varepsilon - (p(x,x_n) - p(x_n,x_n))$. Noting that $p(x,x_n) - p(x_n,x_n) \geq \theta$, it follows that $p(x,x_n) \ll p(x,x) + \varepsilon$. So $x_n \in B(x,\varepsilon)$. This proves that $\text{p-}\lim_{n\to\infty} x_n = x$. $\blacksquare$

PROPOSITION 5.5. *Let (X,p) be a partial tvs-cone metric space and (X,d) be the tvs-cone metric space induced by (X,p). Assume that $\{x_n\}$ is a sequence in X. Then the following are equivalent.*

(1) $\text{d-}\lim_{n\to\infty} x_n = x$.
(2) $\widehat{\lim}_{n\to\infty} d(x,x_n) = \theta$.
(3) $\widehat{\lim}_{n\to\infty} p(x_n,x_n) = \widehat{\lim}_{n\to\infty} p(x_n,x) = p(x,x)$.

Proof. (1)⇒(2): Assume that $\text{d-}\lim_{n\to\infty} x_n = x$. Whenever $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $d(x,x_n) \ll \varepsilon$ for all $n > n_0$, It follows that $\theta - \varepsilon \ll d(x,x_n) \leq \theta + \varepsilon$ for all $n > n_0$. So $\widehat{\lim}_{n\to\infty} d(x,x_n) = \theta$.

(2)⇒(1): Assume that $\widehat{\lim}_{n\to\infty} d(x,x_n) = \theta$. Whenever $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $\theta - \varepsilon \ll d(x,x_n) \ll \theta + \varepsilon$ for all $n > n_0$. It follows that $d(x,x_n) \ll \varepsilon$ for all $n > n_0$. So $\text{d-}\lim_{n\to\infty} x_n = x$.

(2) ⇒ (3): Let $\widehat{\lim}_{n\to\infty} d(x,x_n) = \theta$, i.e., $\widehat{\lim}_{n\to\infty}(2p(x,x_n) - p(x,x) - p(x_n,x_n)) = \theta$. By (2)⇒(1) and Proposition 5.4, $\text{p-}\lim_{n\to\infty} x_n = x$. It follows that $\widehat{\lim}_{n\to\infty} p(x,x_n) = p(x,x)$ from Proposition 3.8. By Lemma 2.10,

$$\begin{aligned}\widehat{\lim_{n\to\infty}} p(x_n,x_n) &= \widehat{\lim_{n\to\infty}}(2p(x,x_n) - p(x,x)) = \widehat{\lim_{n\to\infty}} 2p(x,x_n) - \widehat{\lim_{n\to\infty}} p(x,x)\\ &= 2p(x,x) - p(x,x) = p(x,x).\end{aligned}$$

(3)⇒(2): Let $\widehat{\lim}_{n\to\infty} p(x_n,x_n) = \widehat{\lim}_{n\to\infty} p(x_n,x) = p(x,x)$. By Lemma 2.10,

$$\begin{aligned}\widehat{\lim_{n\to\infty}} d(x,x_n) &= \widehat{\lim_{n\to\infty}}(2p(x,x_n) - p(x,x) - p(x_n,x_n))\\ &= \widehat{\lim_{n\to\infty}} 2p(x,x_n) - \widehat{\lim_{n\to\infty}} p(x,x) - \widehat{\lim_{n\to\infty}} p(x_n,x_n)\\ &= 2p(x,x) - p(x,x) - p(x,x) = \theta. \ \blacksquare\end{aligned}$$

However, we do not know whether Proposition 5.4 can be reversed (see Question 5.10(3)). Now we discuss completeness of (X,p) and (X,d).

DEFINITION 5.6 ([3]). Let (X,d) be a tvs-cone metric space and $\{x_n\}$ be a sequence in X.

(1) $\{x_n\}$ is called a Cauchy sequence in (X,d) if $\widehat{\lim}_{n,m\to\infty} d(x_n,x_m) = \theta$ i.e., whenever $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $d(x_n,x_m) \ll \varepsilon$ for all $n,m > n_0$.

(2) (X,d) is said to be *complete* if every Cauchy sequence in (X,d) is convergent in (X,d).

It is easy to see that every convergent sequence in a tvs-cone metric space (X,d) is a Cauchy sequence in (X,d). However, the following question is still open.

QUESTION 5.7. *Let (X,p) be a partial tvs-cone metric (resp. partial metric) space. Is every convergent sequence in (X,p) a Cauchy sequence in (X,p)?*

For completeness of partial metric spaces, the following is known.

PROPOSITION 5.8 ([1, 5]). *Let (X, p) be a partial metric space and (X, d) be the metric space induced by (X, p).*

(1) *A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if $\{x_n\}$ is a Cauchy sequence in (X, d).*
(2) *(X, p) is complete if and only if (X, d) is complete.*

However, we have only the following results for completeness of partial tvs-cone metric spaces.

PROPOSITION 5.9. *Let (X, p) be a partial tvs-cone metric space and (X, d) be the tvs-cone metric space induced by (X, p).*

(1) *If $\{x_n\}$ is a Cauchy sequence in (X, p), then $\{x_n\}$ is a Cauchy sequence in (X, d).*
(2) *If (X, d) is complete, then (X, p) is complete.*

Proof. (1) Let $\{x_n\}$ be a Cauchy sequence in (X, p). Then

$$\widehat{\lim_{n,m\to\infty}} p(x_n, x_m) = \alpha \quad \text{for some } \alpha \in E.$$

It follows that

$$\begin{aligned}\widehat{\lim_{n,m\to\infty}} d(x_n, x_m) &= \widehat{\lim_{n,m\to\infty}} (2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m)) \\ &= 2\alpha - \alpha - \alpha = \theta.\end{aligned}$$

So $\{x_n\}$ is a Cauchy sequence in (X, d).

(2) Let (X, d) is complete. If $\{x_n\}$ is a Cauchy sequence in (X, p), then it is a Cauchy sequence in (X, d) by (1). Since (X, d) is complete, $\{x_n\}$ is a convergent sequence in (X, d). By Proposition 5.4, $\{x_n\}$ is a convergent sequence in (X, p). So (X, p) is complete. ∎

In view of Proposition 5.9, we raise the following question.

QUESTION 5.10. *Let (X, p) be a partial tvs-cone metric space and (X, d) be the tvs-cone metric space induced by (X, p).*

(1) *Is every Cauchy sequence in (X, d) a Cauchy sequence in (X, p)?*
(2) *Does completeness of (X, p) imply completeness of (X, d)?*
(3) *Can "$p(x, x) = \widehat{\lim}_{n\to\infty} p(x, x_n) = \widehat{\lim}_{n,m\to\infty} p(x_n, x_m)$" in Definition 4.6(2) be replaced by "$p(x, x) = \widehat{\lim}_{n\to\infty} p(x, x_n)$" or "$p(x, x) = \widehat{\lim}_{n,m\to\infty} p(x_n, x_m)$"?*

It is easy to see that the fixed point is unique for a single valued contraction on a complete partial tvs-cone metric space. So we raise the following question to end this paper.

QUESTION 5.11. *Is the fixed point in Theorem 4.9 unique?*

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