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Completions of partial metric spaces

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ABSTRACT

This paper gives the existence and uniqueness theorems in the classical sense for completions of partial metric spaces.

1. Introduction

Partial metric spaces were introduced and investigated by S. Matthews in [10] (also see [3]). In the past years, partial metric spaces had aroused popular attentions and many interesting results are obtained (for example, see [1,2,6,7,9,11]). It is well known that every metric space has a unique completion in the classical sense [5]. But, we do not know if there are similar completion theorems for partial metric spaces. In their paper [8], R. Kopperman, S. Matthews and H. Pajoohesh investigated some notions of completion of partial metric spaces, including the bicompletion, the Smyth completion, and a new "spherical completion". However, completion problem in the classical sense for partial metric spaces is still open. Indeed, it is the most difference from metric that self-distances of some points in partial metric spaces may not be zero, and then $y \in B(x, \varepsilon)$ and $x \in B(y, \varepsilon)$ are not equivalent in general for "ball-neighborhoods" $B(x, \varepsilon)$ and $B(y, \varepsilon)$, which generates many difficulties in investigating partial metric spaces by metric methods.

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In this paper, we introduce symmetrically dense subsets of partial metric spaces to prove the existence and uniqueness theorems in the classical sense for completions of partial metric spaces.

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}^*$ denote the set of all natural numbers and the set of all nonnegative real numbers, respectively.

2. Preliminaries

Definition 1 *([3])*. Let X be a non-empty set. A mapping $p : X \times X \longrightarrow R^*$ is called a partial metric and (X, p) is called a partial metric space if the following are satisfied for all $x, y, z \in X$.

(1) $x = y \Longleftrightarrow p(x, x) = p(y, y) = p(x, y)$.
(2) $p(x, y) = p(y, x)$.
(3) $p(x, x) \leq p(x, y)$.
(4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Remark 1. Let (X, p) be a partial metric space, $x \in X$ and $\varepsilon > 0$. Put $B(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ and put $\mathscr{B} = \{B(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$. Then $\mathscr{B}$ is a base for some topology τ on X ([3]). In this paper, the partial metric space (X, p) is always a topological space (X, τ).

Definition 2 *([3])*. Let (X, p) be a partial metric space.

(1) A sequence $\{x_n\}$ in X is called to be a Cauchy sequence if there is $r \in \mathbb{R}^*$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = r$.

(2) A sequence $\{x_n\}$ in X is called to converge in (X, p) if there is $x \in X$ such that $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x_n)$.

(3) (X, p) is called to be complete if every Cauchy sequence in X converges in (X, p).

Remark 2. Let (X, p) be a partial metric space. In this paper, a convergent sequence $\{x_n\}$ in X always means that $\{x_n\}$ converges in (X, p), which is different from that $\{x_n\}$ converges in (X, τ). It is also worth noting that if $\{x_n\}$ converges to x in (X, p), then $\{x_n\}$ converges to x in (X, τ), but the other direction only yields $p(x, x) = \lim_{n\to\infty} p(x, x_n)$ and $\limsup_{n\to\infty} p(x_n, x_n) \leq p(x, x)$.

The following two definitions adopt the descriptions on "isometry" and "dense" for metric case in [5], respectively.

Definition 3. Let (X, p) and (Y, q) be partial metric spaces. A mapping $f : X \longrightarrow Y$ is called to be an isometry if $q(f(x), f(x')) = p(x, x')$ for all $x, x' \in X$.

Definition 4. Let (X, p) be a partial metric space. A complete partial metric space (X^*, p^*) is called a completion of (X, p) if there is an isometry $f : X \longrightarrow X^*$ such that $f(X)$ is dense in (X^*, p^*).

"Sequentially dense" in topological spaces was introduced by S. Davis in [4], we introduce "sequentially dense" in partial metric spaces as follows.

Definition 5. Let (X, p) be a partial metric space and Y be a subset of X.

(1) Y is called to be sequentially dense in X if for any $x \in X$ there is a sequence in Y converging to x.
(2) Y is called to be symmetrically dense in X if for any $x \in X$ and any $\varepsilon > 0$, there is $y \in Y$ such that $y \in B(x, \varepsilon)$ and $x \in B(y, \varepsilon)$.

Remark 3. It is clear that symmetrically dense and dense are equivalent in metric spaces and symmetrically dense implies dense in partial metric spaces.

Now we give some lemmas for following sections.

Lemma 1. *Let (X, p) be a partial metric space.*

(1) *If $\{x_n\}$ is a convergent sequence in X, then $\{x_n\}$ is a Cauchy sequence.*
(2) *If $\{x_n\}$ is a Cauchy sequence in X and has a convergent subsequence $\{x_{n_i}\}$, then $\{x_n\}$ converges.*
(3) *If $\{x_n\}$ is a sequence in X converging to both x and y, then $x = y$.*
(4) *If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X, then $\lim_{n\to\infty} p(x_n, y_n)$ exists.*
(5) *If $\{x_n\}$ and $\{y_n\}$ are sequences in X converging to x and y respectively, then $\lim_{n\to\infty} p(x_n, y_n) = p(x, y)$.*

Proof. (1) Let $\{x_n\}$ be a sequence in X converging to x. Then $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x_n)$. For $n, m \in \mathbb{N}$, $p(x_n, x_m) \le p(x_n, x) + p(x, x_m) - p(x, x)$ and $p(x, x) \le p(x, x_n) + p(x_n, x_m) + p(x_m, x) - p(x_n, x_n) - p(x_m, x_m)$. Let $n, m \to \infty$. Then $p(x, x) \le \lim_{n,m\to\infty} p(x_n, x_m) \le p(x, x)$. So $\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$, and then $\{x_n\}$ is a Cauchy sequence.

(2) Let $\{x_n\}$ be a Cauchy sequence in X and have a convergent subsequence $\{x_{n_i}\}$. Then there is $x \in X$ such that $p(x, x) = \lim_{i\to\infty} p(x, x_{n_i}) = \lim_{i\to\infty} p(x_{n_i}, x_{n_i})$. Since $\{x_n\}$ is a Cauchy sequence, there is $r \in \mathbb{R}^*$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = r$. It is clear that $p(x, x) = r$. Note that $p(x_n, x) \le p(x_n, x_{n_i}) + p(x_{n_i}, x) - p(x_{n_i}, x_{n_i})$ and $p(x_{n_i}, x) \le p(x_{n_i}, x_n) + p(x_n, x) - p(x_n, x_n)$. So $p(x_{n_i}, x) - p(x_{n_i}, x_n) + p(x_n, x_n) \le p(x_n, x) \le p(x_n, x_{n_i}) + p(x_{n_i}, x) - p(x_{n_i}, x_{n_i})$. Let $n, i \to \infty$. Then $p(x, x) - r + r \le \lim_{n\to\infty} p(x_n, x) \le r + p(x, x) - p(x, x)$. It follows that $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x_n)$. So $\{x_n\}$ converges.

(3) Let the sequence $\{x_n\}$ in X converge to both x and y. Then $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x_n)$ and $p(y, y) = \lim_{n\to\infty} p(y, x_n) = \lim_{n\to\infty} p(x_n, x_n)$. So $p(x, x) = p(y, y)$. Since $p(x, y) \le p(x, x_n) + p(x_n, y) - p(x_n, x_n)$. Let $n \to \infty$. Then $p(x, y) \le p(x, x) + p(y, y) - p(y, y) = p(x, x)$. Since $p(x_n, x_n) \le p(x_n, x) + p(x, y) + p(y, x_n) - p(x, x) - p(y, y)$. Let $n \to \infty$. Then $p(x, x) \le p(x, x) + p(x, y) + p(x, x) - p(x, x) - p(y, y) = p(x, y)$. It follows that $p(x, x) = p(y, y) = p(x, y)$. So $x = y$.

(4) Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in X. Then there are $r, t \in \mathbb{R}^*$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = r$ and $\lim_{n,m\to\infty} p(y_n, y_m) = t$. It suffices to prove that $\{p(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$. For $n, m \in \mathbb{N}$, we have $p(x_n, y_n) \le p(x_n, x_m) + p(x_m, y_m) + p(y_m, y_n) - p(x_m, x_m) - p(y_m, y_m)$ and $p(x_m, y_m) \le p(x_m, x_n) + p(x_n, y_n) + p(y_n, y_m) - p(x_n, x_n) - p(y_n, y_n)$. It follows that $p(x_n, x_n) + p(y_n, y_n) - p(x_m, x_n) - p(y_n, y_m) \le p(x_n, y_n) - p(x_m, y_m) \le p(x_n, x_m) + p(y_m, y_n) - p(x_m, x_m) - p(y_m, y_m)$. Let $n, m \to \infty$. Then $r + t - r - t \le \lim_{n,m\to\infty}(p(x_n, y_n) - p(x_m, y_m)) \le r + t - r - t$. So $\lim_{n,m\to\infty} |p(x_n, y_n) - p(x_m, y_m)| = 0$. This has proved that $\{p(x_n, y_n)\}$ is a Cauchy sequences in $\mathbb{R}$.

(5) Let $\{x_n\}$ and $\{y_n\}$ be sequences in (X, p) converging to x and y respectively. Then $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x_n)$ and $p(y, y) = \lim_{n\to\infty} p(y, y_n) = \lim_{n\to\infty} p(y_n, y_n)$. For $n \in \mathbb{N}$, we have $p(x_n, y_n) \le p(x_n, x) + p(x, y) + p(y, y_n) - p(x, x) - p(y, y)$ and $p(x, y) \le p(x, x_n) + p(x_n, y_n) + p(y_n, y) - p(x_n, x_n) - p(y_n, y_n)$. Let $n \to \infty$. Then $\lim_{n\to\infty} p(x_n, y_n) \le p(x, x) + p(x, y) + p(y, y) - p(x, x) - p(y, y) = p(x, y)$ and $p(x, y) \le p(x, x) + \lim_{n\to\infty} p(x_n, y_n) + p(y, y) - p(x, x) - p(y, y) = \lim_{n\to\infty} p(x_n, y_n)$. So $\lim_{n\to\infty} p(x_n, y_n) = p(x, y)$. □

Lemma 2. *Let $f : X \longrightarrow Y$ be an isometry, where (X, p) and (Y, q) are partial metric spaces. If $\{x_n\}$ is a sequence in X converging to x, then $\{f(x_n)\}$ is a sequence in Y converging to $f(x)$.*

Proof. Let $\{x_n\}$ be a sequence in X converging to x. Then $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x_n)$. Since $f : X \longrightarrow Y$ is an isometry, $q(f(x), f(x)) = \lim_{n\to\infty} q(f(x), f(x_n)) = \lim_{n\to\infty} q(f(x_n), f(x_n))$. So $\{f(x_n)\}$ converges to $f(x)$. □

Lemma 3. *Let (X, p) be a partial metric space and Y be a subset of X. Then the following are equivalent.*

(1) *Y is sequentially dense in X.*
(2) *Y is symmetrically dense in X.*

Proof. (1) $\Longrightarrow$ (2): Let Y be sequentially dense in X and $x \in X$. Then there is a sequence $\{x_n\}$ in Y converges to x, i.e., $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x_n)$. For any $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that for all $n > k$, $|p(x, x_n) - p(x, x)| < \varepsilon/2$ and $|p(x_n, x_n) - p(x, x)| < \varepsilon/2$, hence $p(x, x_n) < p(x, x) + \varepsilon/2 < p(x, x) + \varepsilon$ and $p(x, x_n) < p(x, x) + \varepsilon/2 < p(x_n, x_n) + \varepsilon/2 + \varepsilon/2 = p(x_n, x_n) + \varepsilon$. Pick $n_0 > k$ and put $y = x_{n_0} \in Y$. Then $y \in B(x, \varepsilon)$ and $x \in B(y, \varepsilon)$. So Y is symmetrically dense in X.

(2) $\Longrightarrow$ (1): Let Y be symmetrically dense in X and $x \in X$. Then, for each $n \in \mathbb{N}$, there is $x_n \in Y$ such that $x_n \in B(x, 1/n)$ and $x \in B(x_n, 1/n)$, i.e., $p(x, x_n) < p(x, x) + 1/n$ and $p(x, x_n) < p(x_n, x_n) + 1/n$. It suffices to prove that the sequence $\{x_n\}$ converges to x. Indeed, $p(x, x) \le p(x, x_n) < p(x, x) + 1/n$ for each $n \in \mathbb{N}$. Let $n \to \infty$. Then $\lim_{n\to\infty} p(x, x_n) = p(x, x)$. On the other hand, $p(x, x_n) < p(x_n, x_n) + 1/n \le p(x, x_n) + 1/n$. Let $n \to \infty$. Then $\lim_{n\to\infty} p(x_n, x_n) = p(x, x)$. So the sequence $\{x_n\}$ converges to x. □

Lemma 4. *Let (X, p) be a partial metric space and Y be symmetrically dense in X such that every Cauchy sequence in Y converges in X. Then (X, p) is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Then there is $r \in \mathbb{R}^*$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = r$. For each $i \in \mathbb{N}$, there is $n_i \in \mathbb{N}$ such that $r - 1/i < p(x_n, x_m) < r + 1/i$ for all $n, m \ge n_i$. Without loss of generality, we assume $n_1 < n_2 < \cdots < n_i < \cdots$. Since Y is symmetrically dense in X, for each $i \in \mathbb{N}$, there is $y_i \in Y$ such that $y_i \in B(x_{n_i}, 1/i)$ and $x_{n_i} \in B(y_i, 1/i)$, i.e., $p(x_{n_i}, y_i) < p(x_{n_i}, x_{n_i}) + 1/i$ and $p(x_{n_i}, y_i) < p(y_i, y_i) + 1/i$. We claim that the sequence $\{y_i\}$ is a Cauchy sequence in Y. In fact, for any $\varepsilon > 0$, pick $k \in \mathbb{N}$ such that $1/k < \varepsilon/3$. If $i, j > k$, then $p(y_i, y_j) \le p(y_i, x_{n_i}) + p(x_{n_i}, x_{n_j}) + p(x_{n_j}, y_j) - p(x_{n_i}, x_{n_i}) - p(x_{n_j}, x_{n_j}) < r + 3/k < r + \varepsilon$. On the other hand, $r - 1/k < p(x_{n_i}, x_{n_j}) \le p(x_{n_i}, y_i) + p(y_i, y_j) + p(y_j, x_{n_j}) - p(y_i, y_i) - p(y_j, y_j) \le p(y_i, y_j) + 2/k$. So $r - 3/k < p(y_i, y_j)$, i.e., $r - \varepsilon < p(y_i, y_j)$. This has proved that $\lim_{i,j\to\infty} p(y_i, y_j) = r$. So $\{y_i\}$ is a Cauchy sequence in Y. Since every Cauchy sequence in Y converges in X, $\{y_i\}$ converges in X, i.e., there is $x \in X$ such that $p(x, x) = \lim_{i\to\infty} p(x, y_i) = \lim_{i\to\infty} p(y_i, y_i)$. It is clear that $p(x, x) = r$. For each $i \in \mathbb{N}$, $p(x, x_{n_i}) \le p(x, y_i) + p(y_i, x_{n_i}) - p(y_i, y_i) < p(x, y_i) + 1/i$ and $p(x, y_i) \le p(x, x_{n_i}) + p(x_{n_i}, y_i) - p(x_{n_i}, x_{n_i}) < p(x, x_{n_i}) + 1/i$, so $p(x, y_i) - 1/i < p(x, x_{n_i}) < p(x, y_i) + 1/i$. Let $i \to \infty$. Then $\lim_{i\to\infty} p(x, x_{n_i}) = p(x, x)$. On the other hand, $\lim_{i\to\infty} p(x_{n_i}, x_{n_i}) = r = p(x, x)$. So $\{x_{n_i}\}$ converges. By Lemma 1(2), $\{x_n\}$ converges. This has proved that (X, p) is complete. □

3. The existence

In this section, we give the existence theorem of completions for partial metric spaces, which is proved by six facts.

Theorem 1. *Every partial metric space has a completion.*

Let (X, p) be a partial metric space. Put

$$\mathscr{K} = \big\{\{x_n\} : \{x_n\} \text{ is a Cauchy sequence in } (X, p)\big\}.$$

We define a relation $\sim$ on $\mathscr{K}$ as follows: for $\{x_n\}, \{y_n\} \in \mathscr{K}$, $\{x_n\} \sim \{y_n\} \iff \lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(y_n, y_n) = \lim_{n\to\infty} p(x_n, y_n)$.

Fact 1. The relation $\sim$ is an equivalent relation.

Proof. It is clear that the relation $\sim$ satisfies reflexivity and symmetry. It suffices to prove the relation $\sim$ satisfies transitivity. If $\{x_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$, then $\lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(y_n, y_n) = \lim_{n\to\infty} p(x_n, y_n)$ and $\lim_{n\to\infty} p(y_n, y_n) = \lim_{n\to\infty} p(z_n, z_n) = \lim_{n\to\infty} p(y_n, z_n)$. It follows that $\lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(z_n, z_n)$. Since $p(x_n, z_n) \le p(x_n, y_n) + p(y_n, z_n) - p(y_n, y_n)$, $\lim_{n\to\infty} p(x_n, z_n) \le \lim_{n\to\infty}(p(x_n, y_n) + p(y_n, z_n) - p(y_n, y_n))$, and then $\lim_{n\to\infty} p(x_n, z_n) \le \lim_{n\to\infty} p(x_n, y_n)$. Similarly, $\lim_{n\to\infty} p(x_n, y_n) \le \lim_{n\to\infty} p(x_n, z_n)$. Thus, $\lim_{n\to\infty} p(x_n, y_n) = \lim_{n\to\infty} p(x_n, z_n)$. This has proved that $\lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(z_n, z_n) = \lim_{n\to\infty} p(x_n, z_n)$. So $\{x_n\} \sim \{z_n\}$. □

Let X^* be the set of all equivalence classes in $\mathscr{K}$ for $\sim$: $X^* = \{[\{x_n\}] : \{x_n\} \in \mathscr{K}\}$. Define $p^* : X^* \times X^* \longrightarrow \mathbb{R}^*$ as follows: for $[\{x_n\}], [\{y_n\}] \in X^*$, $p^*([\{x_n\}], [\{y_n\}]) = \lim_{n\to\infty} p(x_n, y_n)$.

Fact 2. p^* is well-defined.

Proof. For $\{x_n\}, \{y_n\} \in \mathscr{K}$, $\lim_{n\to\infty} p(x_n, y_n)$ exists from Lemma 1(4). In addition, let $\{x_n\}, \{y_n\}, \{x'_n\}, \{y'_n\} \in \mathscr{K}$ such that $\{x_n\} \sim \{x'_n\}$ and $\{y_n\} \sim \{y'_n\}$. Then $\lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(x'_n, x'_n) = \lim_{n\to\infty} p(x_n, x'_n)$ and $\lim_{n\to\infty} p(y_n, y_n) = \lim_{n\to\infty} p(y'_n, y'_n) = \lim_{n\to\infty} p(y_n, y'_n)$. Since $p(x_n, y_n) \le p(x_n, x'_n) + p(x'_n, y'_n) + p(y'_n, y_n) - p(x'_n, x'_n) - p(y'_n, y'_n)$, $\lim_{n\to\infty} p(x_n, y_n) \le \lim_{n\to\infty}(p(x_n, x'_n) + p(x'_n, y'_n) + p(y'_n, y_n) - p(x'_n, x'_n) - p(y'_n, y'_n))$, and then $\lim_{n\to\infty} p(x_n, y_n) \le \lim_{n\to\infty} p(x'_n, y'_n)$. Similarly, $\lim_{n\to\infty} p(x'_n, y'_n) \le \lim_{n\to\infty} p(x_n, y_n)$. So $\lim_{n\to\infty} p(x_n, y_n) = \lim_{n\to\infty} p(x'_n, y'_n)$. Consequently, p^* is well-defined. □

Fact 3. p^* is a partial metric on X^*.

Proof. The fact is proved by the following (1) $\sim$ (4).

(1) Let $[\{x_n\}], [\{y_n\}] \in X^*$. Clearly, if $[\{x_n\}] = [\{y_n\}]$, then $p^*([\{x_n\}], [\{x_n\}]) = p^*([\{y_n\}], [\{y_n\}]) = p^*([\{x_n\}], [\{y_n\}])$. Conversely, if $p^*([\{x_n\}], [\{x_n\}]) = p^*([\{y_n\}], [\{y_n\}]) = p^*([\{x_n\}], [\{y_n\}])$, then $\lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(y_n, y_n) = \lim_{n\to\infty} p(x_n, y_n)$, so $\{x_n\} \sim \{y_n\}$, and then $[\{x_n\}] = [\{y_n\}]$.

(2) Let $[\{x_n\}], [\{y_n\}] \in X^*$. Then $p^*([\{x_n\}], [\{y_n\}]) = \lim_{n\to\infty} p(x_n, y_n) = \lim_{n\to\infty} p(y_n, x_n) = p^*([\{y_n\}], [\{x_n\}])$.

(3) Let $[\{x_n\}], [\{y_n\}] \in X^*$. Then $p^*([\{x_n\}], [\{x_n\}]) = \lim_{n\to\infty} p(x_n, x_n) \le \lim_{n\to\infty} p(x_n, y_n) = p^*([\{x_n\}], [\{y_n\}])$.

(4) Let $[\{x_n\}], [\{y_n\}], [\{z_n\}] \in X^*$. Then $p^*([\{x_n\}], [\{z_n\}]) = \lim_{n\to\infty} p(x_n, z_n) \le \lim_{n\to\infty}(p(x_n, y_n) + p(y_n, z_n) - p(y_n, y_n)) = \lim_{n\to\infty} p(x_n, y_n) + \lim_{n\to\infty} p(y_n, z_n) - \lim_{n\to\infty} p(y_n, y_n) = p^*([\{x_n\}], [\{y_n\}]) + p^*([\{y_n\}], [\{z_n\}]) - p^*([\{y_n\}], [\{y_n\}])$. □

For each $x \in X$, let x^* be the equivalence classes of the constant sequence $\{x, x, \cdots,\}$, i.e., $x^* = [\{x, x, \cdots\}] \in X^*$. Define $f : X \longrightarrow X^*$ by $f(x) = x^*$.

Fact 4. f is an isometry from (X, p) into (X^*, p^*).

Proof. Let $x, y \in X$. Then $p^*(f(x), f(y)) = p^*(x^*, y^*) = \lim_{n\to\infty} p(x, y) = p(x, y)$. So f is an isometry from (X, p) into (X^*, p^*). □

Fact 5. $f(X)$ is symmetrically dense in X^*, and then $f(X)$ is dense in X^* from Remark 3.

Proof. Let $[\{x_n\}] \in X^*$ and $\varepsilon > 0$. It suffices to prove that there is $x^* \in f(X)$ such that $x^* \in B([\{x_n\}], \varepsilon)$ and $[\{x_n\}] \in B(x^*, \varepsilon)$. Since $\{x_n\}$ is a Cauchy sequence, there is $r \in \mathbb{R}^*$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = r$. It follows that there is $n_0 \in \mathbb{N}$ such that $r - \varepsilon/3 < p(x_n, x_m) < r + \varepsilon/3$ for all $n, m \ge n_0$. Especially, $r - \varepsilon/3 < p(x_n, x_n) < r + \varepsilon/3$ and $r - \varepsilon/3 < p(x_n, x_{n_0}) < r + \varepsilon/3$ for all $n \ge n_0$. Hence, $r - \varepsilon/3 \le \lim_{n\to\infty} p(x_n, x_n) \le$

$r+\varepsilon/3$, $r-\varepsilon/3 \le \lim_{n\to\infty} p(x_n, x_{n_0}) \le r+\varepsilon/3$ and $r-\varepsilon/3 < p(x_{n_0}, x_{n_0}) < r+\varepsilon/3$. Write $x = x_{n_0}$, then $x^* \in f(X) \subseteq X^*$. Thus, $r-\varepsilon/3 \le p^*([\{x_n\}], [\{x_n\}]) \le r+\varepsilon/3$, $r-\varepsilon/3 \le p^*([\{x_n\}], x^*) \le r+\varepsilon/3$ and $r-\varepsilon/3 < p^*(x^*, x^*) < r+\varepsilon/3$. Consequently, $p^*([\{x_n\}], x^*) \le r+\varepsilon/3 = r-\varepsilon/3+2\varepsilon/3 \le p^*([\{x_n\}], [\{x_n\}]) + 2\varepsilon/3 < p^*([\{x_n\}], [\{x_n\}])+\varepsilon$ and $p^*([\{x_n\}], x^*) \le r+\varepsilon/3 = r-\varepsilon/3+2\varepsilon/3 < p^*(x^*, x^*)+2\varepsilon/3 < p^*(x^*, x^*)+\varepsilon$. So $x^* \in B([\{x_n\}], \varepsilon)$ and $[\{x_n\}] \in B(x^*, \varepsilon)$. □

Fact 6. (X^*, p^*) is complete.

Proof. Let $\{x_n^*\}$ be a Cauchy sequence in $f(X)$, where $x_n^* = [\{x_n, x_n, \cdots\}]$. Then there is $r \in \mathbb{R}^*$ such that $\lim_{n,m\to\infty} p^*(x_n^*, x_m^*) = r$. By Fact 5 and Lemma 4, it suffices to prove that $\{x_n^*\}$ converges in (X^*, p^*). By Fact 4, f is an isometry. So, for $n, m \in \mathbb{N}$, $p(x_n, x_m) = p^*(f(x_n), f(x_m)) = p^*(x_n^*, x_m^*)$. So $\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n,m\to\infty} p^*(x_n^*, x_m^*) = r$, and then $\{x_n\}$ is a Cauchy sequence in X. Write $\widetilde{x} = [\{x_n\}] \in X^*$. Then $p^*(\widetilde{x}, \widetilde{x}) = \lim_{n,m\to\infty} p(x_n, x_n) = r = \lim_{n\to\infty} p^*(x_n^*, x_n^*)$. On the other hand, for each $n \in \mathbb{N}$, $p^*(x_n^*, \widetilde{x}) = \lim_{k\to\infty} p(x_n, x_k)$. So $\lim_{n\to\infty} p^*(x_n^*, \widetilde{x}) = \lim_{n\to\infty}(\lim_{k\to\infty} p(x_n, x_k)) = \lim_{n,k\to\infty} p(x_n, x_k) = r = p^*(\widetilde{x}, \widetilde{x})$. This has proved that $\{x_n^*\}$ converges. So (X^*, p^*) is complete. □

4. The uniqueness

In this section, we give a uniqueness theorem of completions for partial metric spaces.

Proposition 1. *Let (X^*, p) and (Y^*, q) are two complete partial metric spaces, X and Y be symmetrically dense subsets of X^* and Y^* respectively. If $h : X \longrightarrow Y$ is an isometry, then there is a unique isometry extension $f : X^* \longrightarrow Y^*$, which is an extension of h.*

Proof. Let $h : X \longrightarrow Y$ be an isometry. By Lemma 3, for each $x \in X^*$, there is a sequence $\{x_n\}$ in X converging to x. By Lemma 1(1), $\{x_n\}$ is a Cauchy sequence in X, so $\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$. Since $h : X \longrightarrow Y$ is an isometry, $\lim_{n,m\to\infty} q(h(x_n), h(x_m)) = \lim_{n\to\infty} p(x_n, x_m) = p(x, x)$, hence $\{h(x_n)\}$ is a Cauchy sequence in Y^*. By the completeness of Y^*, $\{h(x_n)\}$ converges to some $y \in Y^*$. Put $f(x) = y$. Thus, we have defined $f : X^* \longrightarrow Y^*$. We complete the proof the proposition by the following four claims.

Claim 1. *f is well defined.*

Let $\{x_n\}$ and $\{x_n'\}$ are sequences in X converging to x. Then $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x_n)$ and $p(x, x) = \lim_{n\to\infty} p(x, x_n') = \lim_{n\to\infty} p(x_n', x_n')$. Similar to discussion on the above, there are $y, y' \in Y^*$ such that $\{h(x_n)\}$ and $\{h(x_n')\}$ converge y and y' in Y^* respectively. Thus, $q(y, y) = \lim_{n,m\to\infty} q(h(x_n), h(x_n)) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x)$ and $q(y', y') = \lim_{n,m\to\infty} q(h(x_n'), h(x_n')) = \lim_{n\to\infty} p(x_n', x_n') = p(x, x)$. On the other hand, by Lemma 1(5), $q(y, y') = \lim_{n,m\to\infty} q(h(x_n), h(x_n')) = \lim_{n\to\infty} p(x_n, x_n') = p(x, x)$. So $q(y, y) = q(y', y') = q(y, y')$. It follows that $y = y'$. This has proved that f is well defined.

Claim 2. *f is an extension of h.*

It is clear.

Claim 3. *f is an isometry.*

Let $x, x' \in X^*$. Then there is sequences $\{x_n\}$ and $\{x_n'\}$ in X converging to x and x' respectively. For $n \in \mathbb{N}$, $q(f(x), f(x')) \le q(f(x), f(x_n)) + q(f(x_n), f(x_n')) + q(f(x_n'), f(x')) - q(f(x_n), f(x_n)) - q(f(x_n'), f(x_n'))$ and $p(x, x') \le p(x, x_n) + p(x_n, x_n') + p(x_n', x') - p(x_n, x_n) - p(x_n', x_n')$. Let $n \to \infty$. Then

$q(f(x), f(x')) \leq \lim_{n\to\infty} q(f(x_n), f(x'_n)) = \lim_{n\to\infty} q(h(x_n), h(x'_n)) = \lim_{n\to\infty} p(x_n, x'_n) = p(x, x')$ and $p(x, x') \leq \lim_{n\to\infty} p(x_n, x'_n) = \lim_{n\to\infty} q(h(x_n), h(x'_n)) = \lim_{n\to\infty} q(f(x_n), f(x'_n)) = q(f(x), f(x'))$. It follows that $q(f(x), f(x')) = p(x, x')$. So f is an isometry.

Claim 4. *Let $g : X^* \longrightarrow Y^*$ is an isometry, which is an extension of h. Then $f = g$.*

Let $x \in X^*$. Then there is a sequence $\{x_n\}$ in X converging to x. By Lemma 2, $\{g(x_n)\}$ converges to $g(x)$, i.e., $\{h(x_n)\}$ converges to $g(x)$. Since $\{h(x_n)\}$ converges to $f(x)$. By Lemma 1(3), $f(x) = g(x)$. This has proved that $f = g$. □

Theorem 2. *The completion of a partial metric space (X, p) is unique with respect to isometry under symmetrical denseness. More precisely, if (X_1^*, p_1^*) and (X_2^*, p_2^*) are two completions of (X, p), then there is a unique subjective isometry $f : X_1^* \longrightarrow X_2^*$ such that $ff_1 = f_2$, where $f_1 : X \longrightarrow X_1^*$ and $f_2 : X \longrightarrow X_2^*$ are isometries, $f_1(X)$ and $f_2(X)$ symmetrically dense in X_1^* and X_2^* respectively.*

Proof. Since f_1 is an isometry, f_1 is 1-1. Thus $f_1^{-1} : f_1(X) \longrightarrow X$ is a subjective isometry. Since $f_2 : X \longrightarrow f(X)$ is a surjective isometry, $f_2 f_1^{-1} : f_1(X) \longrightarrow f_2(X)$ is a surjective isometry. Put $h = f_2 f_1^{-1}$. It is clear that $hf_1 = f_2 : X \longrightarrow X_2^*$. By Proposition 1, there is a unique isometry $f : X_1^* \longrightarrow X_2^*$, which is an extension of h. For each $x \in X$, $ff_1(x) = f(f_1(x)) = hf_1(x) = f_2(x)$, and then $ff_1 = f_2$. Similarly, there is a unique isometry $g : X_2^* \longrightarrow X_1^*$ such that $gf_2 = f_1$. It follows that $(gf)f_1 = g(ff_1) = gf_2 = f_1$ and $(fg)f_2 = f(gf_2) = ff_1 = f_2$. Therefore gf restricting on $f_1(X)$ and fg restricting on $f_2(X)$ are identical mappings. Since $f_1(X)$ and $f_2(X)$ symmetrically dense in X_1^* and X_2^* respectively, gf and fg are identical mappings on X_1^* and X_2^* respectively by Proposition 1, and then $f = g^{-1}$. This has proved that $f : X_1^* \longrightarrow X_2^*$ is a unique subjective isometry such that $ff_1 = f_2$. □

5. Conclusion

Let (X, p) be a partial metric space. The proof of Theorem 1 constructs a completion of (X, p) in which (X, p) is not only dense but also symmetrically dense. On the other hand, Theorem 2 gives a uniqueness theorem for the completion of partial metric spaces with respect to isometry under symmetrical denseness. However, we do not know whenever there exists a completion of (X, p) in which (X, p) is dense, but not symmetrically dense. So the following question is worthy to be considered.

Question 1. For a partial metric space (X, p), can one construct a completion in which (X, p) is dense, but not symmetrically dense?

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