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The extensions of some convergence phenomena in topological groups ☆

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ABSTRACT

In this paper sequentially compact sets, weakly first-countable sets and generalized metric sets in extensions of topological groups are studied. Some three space properties on convergence phenomena are obtained. It is shown that (1) if H is a closed subgroup of a topological group G such that all sequentially compact subsets of both the group H and the quotient space G/H are sequential, then all sequentially compact subsets of G are sequential; (2) let H be a closed and second-countable subgroup of a topological group G, then G is a topological sum of $\aleph_0$-subspaces if the quotient space G/H is a local $\aleph_0$-space; (3) let H be a locally compact and metrizable subgroup of a topological group G, then G is sequential if the quotient space G/H is sequential.

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1. Introduction

One of the main operations on topological groups is that of taking quotient groups. Many non-trivial examples and counterexamples arise as quotients of relatively simple and well-known topological groups. This operation has been the subject of an intensive and thorough study; but there still exists a wealth of interesting open problems related to the behavior of different topological and algebraic properties under taking quotients [2].

The following general question is considered in [1]. Let H be a closed subgroup of a topological group G, and G/H the quotient space. Suppose that both H and G/H belong to some nice class of topological spaces. When can we conclude that G is in the same class? The group G is called an *extension of the group H by the quotient space G/H* [27].

In 1949, J.P. Serre [21] proved that if H is a closed subgroup of a topological group G, and both the spaces H and G/H are locally compact, then the topological group G is locally compact. This classical result on the extension of properties from G/H to G induces a study of the above general question. Suppose that H is a closed subgroup of a topological group G, and G/H is the corresponding quotient space. A (topological, algebraic, or a mixed nature) property $\mathcal{P}$ is said to be a *three space property* [5] if, for every topological group G and a closed subgroup H of G, the fact that both spaces H and G/H have $\mathcal{P}$ implies that G also has $\mathcal{P}$. The Serre's theorem implies that local compactness is a three space property. In fact, compactness, completeness, pseudocompactness, connectedness and metrizability are three space properties in the class of topological groups, but countable compactness is not [4].

Recently, A.V. Arhangel'skiĭ, M. Bruguera, M.G. Tkachenko and V.V. Uspenskij [2–5,25] discovered a series of results on the extensions of topological groups with respect to closed invariant subgroups, locally compact subgroups or locally compact metrizable subgroups. These results show that finding three space properties is one of interesting questions in topological groups. Some problems are still open in this direction.

Question 1.1 *([2, Open problem 1.5.1]). Characterize (or find the typical properties) of compact spaces that can be represented as quotients of topological groups with respect to closed metrizable subgroups.*

Question 1.2 *([2, Open problem 9.10.1]). Suppose that H is a closed invariant subgroup of a topological group G, and all compact subsets of the groups H and G/H are sequentially compact. Does G have the same property?*

Question 1.3 *([2, Open problem 9.10.3]). Let all compact subsets of the groups H and G/H be Fréchet–Urysohn. Does the same hold for compact subsets of G?*

Convergence is a basic research object in general topology. It is natural and quite plausible to expect that certain convergence properties of topological spaces should become stronger in topological groups [22]. The most obvious example of this phenomena is that first-countability becomes equivalent to metrizability in topological groups [2]. Some convergence properties in topological groups were introduced in [2]. In this paper we consider the extensions of some convergence properties in topological groups. In Section 2 the three space question for sequentially compact sets is discussed related to Question 1.2. It is shown that if H is a closed subgroup of a topological group G such that all compact subsets of the group H are first-countable, then all compact subsets of G are Fréchet if so is G/H, which gives a partial answer to Question 1.3. In Section 3 the quotients with respect to second-countable subgroups of topological groups are considered. It is proved that if H is a closed second-countable subgroup of a topological group G, then G is a topological sum of $\aleph_0$-subspaces if G/H is a local $\aleph_0$-space. In Section 4 the quotients with respect to locally compact subgroups of topological groups are studied. It is shown that a topological group G is a sequential space if H a locally compact metrizable subgroup of G and the quotient space G/H is sequential.

All spaces in this paper satisfy the T_2-separation axiom. All mappings are continuous and onto. If H is a closed subgroup of a topological group G, we denote by G/H the set of all left cosets aH of H in G, and endow it with the quotient topology with respect to the canonical mapping $\pi : G \to G/H$ defined by $\pi(a) = aH$, for each $a \in G$. Then the mapping π is open, and G/H is a homogeneous space [2, Theorem 1.5.1]. If H is a closed invariant subgroup of a topological group G, then G/H with the quotient topology and multiplication is a topological group, and the canonical mapping $\pi : G \to G/H$ is an open homomorphism [2, Theorem 1.5.3]. The reader may consult [2,7] for unstated notations and terminology.

2. The three space property for sequentially compact sets

A topological property $\mathcal{P}$ is called an *inverse fiber property* [5] if for any mapping $f : X \to Y$ such that the space Y and the fibers of f have $\mathcal{P}$, the space X also has $\mathcal{P}$. $\mathcal{P}$ is called a *regular-inverse fiber property* if for any mapping $f : X \to Y$ such that the space X is regular, and the space Y and the fibers of f have $\mathcal{P}$, the space X also has $\mathcal{P}$.

Lemma 2.1 *([5]). Every inverse fiber property is a three space property.*

Let $\mathcal{P}$ be a topological property. A space X is called *$\mathcal{P}$-closed* (resp., *$\mathcal{P}$-compact*) if every subset of X with property $\mathcal{P}$ is closed (resp., compact). A space X is called *locally* $\mathcal{P}$ if for every $x \in X$ there exists a neighborhood U of the point x with property $\mathcal{P}$.

Lemma 2.2. *Suppose that $\mathcal{P}$ is a topological property preserved by continuous mappings and also inherited by closed sets. Then the property of being $\mathcal{P}$-closed (resp., $\mathcal{P}$-compact) is a regular-inverse fiber property (resp., inverse fiber property).*

Proof. First, let $f : X \to Y$ be a mapping such that the space X is regular, and the space Y and the fibers of f are $\mathcal{P}$-closed. Suppose to the contrary that C is a non-closed subset of X with property $\mathcal{P}$. Then one can take a point $x \in \overline{C} \setminus C$. The set $K = C \cap f^{-1}(f(x))$ has property $\mathcal{P}$ as a closed subset of C and, since the fiber $f^{-1}(f(x))$ is $\mathcal{P}$-closed, K is closed in X. Since X is regular and $x \notin K$, we can choose an open neighborhood U of x in X such that $\overline{U} \cap K = \emptyset$. Then $D = \overline{U} \cap C$ has property $\mathcal{P}$ as a closed subset of C and $x \in \overline{D} \setminus D$. It follows from our choice of U that $D \cap K = \emptyset$ and $f(x) \in \overline{f(D)} \setminus f(D)$, so $f(D)$ is a non-closed subset of Y with property $\mathcal{P}$. This contradicts our assumption about Y.

Second, let $f : X \to Y$ be a mapping such that the space Y and the fibers of f are $\mathcal{P}$-compact. Let C be a subset of X with property $\mathcal{P}$. Then the image $D = f(C)$ is a subset of Y with property $\mathcal{P}$, so D is compact. In addition, if $y \in D$, then $C_y = C \cap f^{-1}(y)$ is a subset with property $\mathcal{P}$ as a closed subset of C. Since $f^{-1}(y)$ is $\mathcal{P}$-compact, it follows that C_y is compact. So, $g = f|_C : C \to D$ is a mapping with compact fibers. If K is closed in C, then K is a subset of X with property $\mathcal{P}$, so $f(K) = g(K)$ is a subset with property $\mathcal{P}$ in Y, hence $g(K)$ is compact and $g(K)$ is closed in D. It follows that g is a perfect mapping. Since D is compact, we conclude that C is also compact. □

The proof of the following lemma is direct.

Lemma 2.3. *Let $\mathcal{P}$ be a topological property such that*

(1) *the disjoint topological sum of spaces with property $\mathcal{P}$ has property $\mathcal{P}$;*
(2) *$\mathcal{P}$ is inherited by open sets;*
(3) *$\mathcal{P}$ is preserved by continuous open mappings.*

Then a space X is locally $\mathcal{P}$ if and only if it has property $\mathcal{P}$.

Let us recall some concepts related to convergence. Let X be a topological space. A subset A of X is called *sequential closed* if no sequence of points of A converges to a point not in A. X is called *sequential* [9] if each sequentially closed subset of X is closed. A space X is called *Fréchet at a point* $x \in X$ if for every $A \subset X$ with $x \in \overline{A} \subset X$ there is a sequence $\{x_n\}_n$ in A such that $\{x_n\}_n$ converges to x in X. A space X is called *Fréchet* [9] if it is Fréchet at every point $x \in X$. A space X is called *strictly Fréchet at a point* $x \in X$ (resp., *strongly Fréchet at a point* $x \in X$) if whenever $\{A_n\}_n$ is a sequence (resp., decreasing sequence) of subsets in X and $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, there exists $x_n \in A_n$ for each $n \in \mathbb{N}$ such that the sequence $\{x_n\}_n$ converges to x. A space X is called *strictly Fréchet* [10] (resp., *strongly Fréchet* [24]) if it is strictly Fréchet (resp., strongly Fréchet) at every point $x \in X$. Fréchet spaces (resp., strongly Fréchet spaces, strictly Fréchet spaces) are also called Fréchet–Urysohn spaces (resp., strongly Fréchet–Urysohn spaces, strictly Fréchet–Urysohn spaces).

Lemma 2.3 is applicable to the following properties of spaces [11]: first-countable spaces, strictly Fréchet spaces, strongly Fréchet spaces, Fréchet spaces, sequential spaces, and k-spaces.

It is well-known [20] that

(1) every first-countable space is a strictly Fréchet space;
(2) every strictly Fréchet space is a strongly Fréchet space;
(3) every strongly Fréchet space is a Fréchet space;
(4) every Fréchet space is a sequential space;
(5) every sequential space is a k-space.

Lemma 2.4. *If all countably compact (resp., sequentially compact) subsets of a topological space X are sequential, then all countably compact (resp., sequentially compact) subsets of X are closed.*

Proof. Suppose that all countably compact (resp., sequentially compact) subsets of the space X are sequential. Let A be a countably compact (resp., sequentially compact) subset of X. If the set A is not closed in X, take a point $x \in \overline{A} \setminus A$. Put $B = A \cup \{x\}$. Then B is also countably compact (resp., sequentially compact) in X, and B is sequential. Since A is not closed in B, then A is not sequentially closed in B, thus there exists a sequence $\{x_n\}_n$ in A such that the sequence $\{x_n\}_n$ converges to x in B. Since A is countably compact (resp., sequentially compact), the sequence $\{x_n\}_n$ has an accumulation point in A, then x is the unique accumulation point and $x \in A$, which is a contradiction. Hence, A is closed in X. This completes the proof. □

The character of a point x (resp., a subset F) in a topological space X is denoted by $\chi(x, X)$ (resp., $\chi(F, X)$), where $\chi(x, X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a local base at } x \text{ of } X\} + \omega$, $\chi(F, X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a neighborhood base at } F \text{ in } X\} + \omega$. Similarly, the character of a topological space X is denoted by $\chi(X)$.

Lemma 2.5. *First-countability is an inverse fiber property for sequentially compact sets, i.e., the property "all sequentially compact sets are first-countable" is an inverse fiber property.*

Proof. Let $f : X \to Y$ be a mapping. Suppose that all sequentially compact subsets of both the space Y and of the fibers $f^{-1}(y)$ for each $y \in Y$ are first-countable. Let C be a sequentially compact subspace of X. Then the image $K = f(C)$ is sequentially compact and, hence, satisfies $\chi(K) \leq \omega$. Take an arbitrary point $x \in C$ and put $y = f(x)$. Then $\chi(y, K) \leq \omega$. Let $g = f|_C : C \to K$. Since the space K is first-countable and every sequentially compact subset of a first-countable space is closed, g is a closed mapping. The set $C_x = g^{-1}(y) = C \cap f^{-1}(f(x))$ is sequentially compact as a closed subset of C, so $\chi(C_x) \leq \omega$. We have $\chi(g(x), K) \leq \omega$ and $\chi(x, C_x) \leq \omega$, whence it follows that $\chi(x, C) \leq \omega$ by [7, 3.7.F]. This proves that $\chi(C) \leq \omega$, i.e., C is first-countable. □

A topological space X has a *G_δ-diagonal* [12] if the diagonal $\Delta = \{(x,x) : x \in X\}$ of $X \times X$ is a G_δ-set in the product space $X \times X$.

Lemma 2.6. *Let G be a topological group. If every sequentially compact subspace of G is first-countable, then every sequentially compact subspace of G is metrizable.*

Proof. Suppose that X is a non-empty sequentially compact subset of G. Consider the mapping $j : G \times G \to G$ defined by $j(x,y) = x^{-1}y$ for all $x, y \in G$. Clearly, $X \times X$ is sequentially compact and j is continuous, so the image $F = j(X \times X)$ is a sequentially compact subset of G which contains the identity e of G. Then F is first-countable by our assumption, so $\{e\}$ is a G_δ-set in F, then $(j|_{X\times X})^{-1}(e) = \Delta$ is the diagonal in $X \times X$, and Δ is a G_δ-set in $X \times X$, i.e., the sequentially compact space X has a G_δ-diagonal, thus X is metrizable [12, Theorem 2.14]. □

The three space problem for compact (resp., countably compact, pseudocompact) sets was discussed by Bruguera and Tkachenko in [5]. The next theorem gives some results with respect to the three space property for sequentially compact sets.

Theorem 2.7. *Each of the following is a three space property:*

(a) *all sequentially compact subsets are closed;*
(b) *all sequentially compact subsets are compact;*
(c) *all sequentially compact subsets are sequential;*
(d) *all sequentially compact subsets are first-countable;*
(e) *all sequentially compact subsets are metrizable.*

Proof. By Lemmas 2.1 and 2.2, (a) and (b) hold.

Suppose that H is a closed subgroup of a topological group G. Suppose also that all sequentially compact subsets of both the group H and the quotient space G/H are sequential. Then all sequentially compact subsets of H and G/H are closed by Lemma 2.4. It follows from (a) that all sequentially compact subsets of G are closed. Let B be a sequentially compact subset of G, and suppose that A is a sequentially closed subset of B. Then A is also a sequentially compact subset of G, thus A is closed in G, i.e., B is sequential. Hence, all sequentially compact subsets of G are sequential. Thus, (c) holds.

By Lemmas 2.1 and 2.5, (d) holds. By Lemma 2.6 and (d), (e) holds. □

It is worth noting that metrizability of sequentially compact sets is not an inverse fiber property. Indeed, the canonical projection q of the Alexandroff double circle [7, Example 3.1.26] X onto the closed unit circle $\mathbb{S}^1$ is a two-to-one mapping, so the image $\mathbb{S}^1$ of X and the fibers of q are sequentially compact and metrizable, while X is a non-metric sequentially compact space. Hence, Theorem 2.7(e) reflects a special behavior of sequential compactness in topological groups.

A space X has *countable tightness* [20] if for each subset A of X and each point $x \in \overline{A}$, there exists a countable subset C of A such that $x \in \overline{C}$. Countable tightness is a three space property for compact subsets [5, Theorem 2.16(o)].

Question 2.8. *Is countable tightness a three space property for sequentially compact sets, i.e., is the property "all sequentially compact sets are of countable tightness" a three space property?*

In the end of this section we consider Question 1.3.

Lemma 2.9. *If every compact (resp., countably compact, sequentially compact) subspace of a topological group G is Fréchet, then every compact (resp., countably compact, sequentially compact) subspace of G is also strongly Fréchet.*

Proof. First, every compact (resp., countably compact, sequentially compact) subset of the topological group G is closed by our assumption and Lemma 2.4. Let A be a compact (resp., countably compact, sequentially compact) subset of G. Then A is closed and Fréchet. Suppose that $\{A_n\}_n$ is a decreasing sequence of subsets of A with $a \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. We can assume that a is an accumulation point of A. Since A is Fréchet, there exists a sequence $\{a_n\}_n$ in $A \setminus \{a\}$ converging to a. Put

$$B = a^{-1}A, \quad \text{and} \quad B_n = a^{-1}A_n, \qquad b_n = a^{-1}a_n \quad \text{for each } n \in \mathbb{N}.$$

Then the set B is closed in G. Let e be the neutral element of the group G. Then $e \in a^{-1}\overline{A_n} = \overline{B_n} \subset B$, $b_n \in B \setminus \{e\}$ for each $n \in \mathbb{N}$, and the sequence $\{b_n\}_n$ converges to e. There exists a sequence $\{V_n\}_n$ of symmetric open neighborhoods of e in G with $b_n \notin V_n^2$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, put $C_n = b_n(B_n \cap V_n)$. Then $b_n \in \overline{C_n}$ for $e \in \overline{B_n \cap V_n}$, while $e \notin \overline{C_n}$ since $V_n \cap C_n \subset V_n \cap b_nV_n = \emptyset$. Put

$$D = \bigcup\{C_n : n \in \mathbb{N}\}, \quad \text{and} \quad S = \{e\} \cup \{b_n : n \in \mathbb{N}\}.$$

Then $D \subset \bigcup_{n\in\mathbb{N}} b_nB_n \subset SB$.

Next, we shall show that the subspace SB of G is closed and Fréchet. Obviously, S is compact and sequentially compact. Since the set A is compact (resp., countably compact, sequentially compact), then B is also compact (resp., sequentially compact, because a Fréchet countably compact space is sequentially compact), thus the Cartesian product $S \times B$ of the spaces S and B is compact (resp., sequentially compact). Since the multiplication in G is jointly continuous and the subset SB of G is the continuous image of the subset $S \times B$ of $G \times G$ under the multiplication mapping, SB is compact (resp., sequentially compact). Thus SB is closed and Fréchet by our assumption.

Since $b_n \in \overline{C_n}$ for each $n \in \mathbb{N}$ and $b_n \to e$, then $e \in \overline{D} \subset SB$, and there is a sequence $\{d_k\}_k$ in D converging to e. For each $n \in \mathbb{N}$, since $e \notin \overline{C_n}$, C_n contains only finitely many terms of the sequence $\{d_k\}_k$. There is a subsequence $\{C_{n_k}\}_k$ of the sequence $\{C_n\}_n$ such that $d_k \in C_{n_k}$ for each $k \in \mathbb{N}$. It follows from $C_{n_k} \subset b_{n_k}B_{n_k} = b_{n_k}a^{-1}A_{n_k}$ that $d_k = b_{n_k}a^{-1}x_{n_k}$ for some $x_{n_k} \in A_{n_k}$ for each $k \in \mathbb{N}$. Then $x_{n_k} = a(b_{n_k})^{-1}d_k \to a$ whenever $k \to \infty$. Take $y_n = x_{n_k}$ when $n_{k-1} < n \leqslant n_k$, then $y_n \in A_n$ for each $n \in \mathbb{N}$ and $y_n \to a$. Hence, A is strongly Fréchet. $\square$

Lemma 2.10 *([3, Proposition 2.18]). Suppose that X is a regular space, and that $f : X \to Y$ is a closed mapping. Suppose also that $b \in X$ is a G_δ-point in the space $F = f^{-1}(f(b))$ (i.e., the singleton $\{b\}$ is a G_δ-set in the space F) and F is Fréchet at b. If the space Y is strongly Fréchet, then X is Fréchet at b.*

Lemma 2.11. *Suppose that X is a regular space, and that $f : X \to Y$ is a closed mapping. Suppose also that $b \in X$ is a G_δ-point in the space $F = f^{-1}(f(b))$ and F is countably compact and strictly Fréchet at b. If the space Y is strictly Fréchet at $f(b)$, then X is strictly Fréchet at b.*

Proof. Using the regularity of the topological space X, we can construct in a standard way a sequence $\{U_n\}_n$ of open subsets in X such that $\{b\} = F \cap \bigcap_{n\in\mathbb{N}} U_n$, and $\overline{U_{n+1}} \subset U_n$ for each $n \in \mathbb{N}$.

Let $\{A_n\}_n$ be a sequence of subsets in X and $b \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. For each $n \in \mathbb{N}$, put $B_n = A_n \cap U_n$ and $C_n = f(B_n)$. Clearly, $b \in \overline{B_n}$ and, therefore, the continuity of f implies that $c = f(b) \in \overline{C_n}$. Since Y is strictly Fréchet at c, there exists $y_n \in C_n$ for each $n \in \mathbb{N}$ such that the sequence $\{y_n\}_n$ converges to c. For each $n \in \mathbb{N}$, fix $x_n \in B_n \subset A_n$ with $f(x_n) = y_n$. We claim that the sequence $\{x_n\}_n$ converges to b, which implies X is strictly Fréchet at b.

First, we show that each subsequence of the sequence $\{x_n\}_n$ in X has an accumulation point in the countably compact set F. Let $\{x_{n_k}\}_k$ be a subsequence of $\{x_n\}_n$. Then $f(x_{n_k}) \to c$. If $\{f(x_{n_k}) : k \in \mathbb{N}\}$ is a finite set, we can assume that $f(x_{n_k}) = c$ for each $k \in \mathbb{N}$. Since F is countably compact, the sequence $\{x_{n_k}\}_k$ in X has an accumulation point in F. If $\{f(x_{n_k}) : k \in \mathbb{N}\}$ is an infinite set, then the set $\{f(x_{n_k}) : k \in \mathbb{N}\}$ is not closed discrete in Y. Since f is closed, the sequence $\{x_{n_k}\}_k$ has an accumulation point in F. It follows that each subsequence of $\{x_n\}_n$ has an accumulation point in F. Next, take an arbitrary point $z \in F$ distinct from b. There exists $m \in \mathbb{N}$ such that $z \notin \overline{U_m}$. Since $\{x_n : n > m\} \subset U_m$, it follows that z cannot be an accumulation point of any subsequence of $\{x_n\}_n$. Thus, b is the unique accumulation point of every subsequence of $\{x_n\}_n$, which implies that $\{x_n\}_n$ converges to b. □

Theorem 2.12. *Suppose that H is a closed subgroup of a topological group G such that all compact subsets (resp., countably compact, sequentially compact) of the group H are first-countable. If the quotient space G/H has one of the following properties, then so does the group G:*

(a) *all compact (resp., countably compact, sequentially compact) subsets are strongly Fréchet;*
(b) *all compact (resp., countably compact, sequentially compact) subsets are strictly Fréchet.*

In addition, if H is an invariant subgroup of the group G and the quotient group G/H has one of the following properties, then so does the group G:

(c) *all compact (resp., countably compact, sequentially compact) subsets are Fréchet.*

Proof. It is well-known that every T_2 topological group is regular. Let C be a compact (resp., countably compact, sequentially compact) subset of the topological group G. By Lemmas 2.1, 2.2 and 2.4, the set C is closed in G. Put $f = \pi|_C : C \to \pi(C)$. Then $\pi(C)$ is compact (resp., countably compact, sequentially compact). It follows from Lemma 2.4 that f is a closed mapping, and $f^{-1}(f(b)) = \pi^{-1}(\pi(b)) \cap C = bH \cap C$ is first-countable for each $b \in C$. By Lemmas 2.9, 2.10 and 2.11, the required conclusions follows. □

It is also worth noting that the property "all compact sets are Fréchet" is not an inverse fiber property. Indeed, Simon [23] constructed a compact Fréchet space X such that X^2 is not Fréchet. It is easy to see that the product space X^2 is compact and sequentially compact.

3. Quotients with respect to second-countable subgroups

In this section we consider extensions of topological groups with certain networks. Being separable and metrizable is a three space property [2, Corollary 3.3.21]. Every separable metrizable space is a cosmic space, i.e., a regular space with a countable network. But, being a cosmic space is not a three space property [27]. Under certain additional conditions on a closed subgroup of a topological group we can obtain some new extension properties of topological groups.

Let $\mathcal{P}$ be a family of subsets of a topological space X. $\mathcal{P}$ is called a *network* [7] for X if whenever $x \in U$ with U open in X, then there exists $P \in \mathcal{P}$ such that $x \in P \subset U$. $\mathcal{P}$ is called a *cs-network* [14] for X if, given a sequence $\{x_n\}_n$ converging to a point x in X and a neighborhood U of x in X, then $\{x\} \cup \{x_n : n \geq n_0\} \subset P \subset U$ for some $n_0 \in \mathbb{N}$ and some $P \in \mathcal{P}$. A space is an $\aleph_0$*-space* if it is a regular space having a countable *cs*-network [14].

It is obvious that [12]:

(1) every separable metric space is an $\aleph_0$-space;
(2) every $\aleph_0$-space is a cosmic space;
(3) every cosmic space is a paracompact, separable space.

Lemma 3.1 *([2, Theorem 1.5.23]). Suppose that G is a topological group and H is a closed and separable subgroup of G. If Y is a separable subset of G/H, then $\pi^{-1}(Y)$ is also separable in G.*

Lemma 3.2 *([3, Corollary 1.2]). A locally paracompact topological group is paracompact.*

A family $\mathcal{P}$ of subsets of a topological space X is called *star-countable* [7] if the collection $\{P \in \mathcal{P} : P \cap P_0 \neq \emptyset\}$ is countable for any $P_0 \in \mathcal{P}$.

Lemma 3.3 *([6, Lemma 3.10]). Every star-countable family $\mathcal{P}$ of subsets of a topological space X can be expressed as $\mathcal{P} = \bigcup\{\mathcal{P}_\alpha : \alpha \in \Lambda\}$ where each subfamily $\mathcal{P}_\alpha$ is countable and $(\bigcup \mathcal{P}_\alpha) \cap (\bigcup \mathcal{P}_\beta) = \emptyset$ whenever $\alpha \neq \beta$.*

Theorem 3.4. *Let H be a closed second-countable subgroup of a topological group G. If the quotient space G/H is a local $\aleph_0$-space (resp., locally cosmic space), then G is a topological sum of $\aleph_0$-subspaces (resp., cosmic subspaces).*

Proof. We need only to consider the case of $\aleph_0$-spaces, the case of cosmic spaces is similar.

Suppose that the quotient space G/H is a local $\aleph_0$-space. Then there exists an open neighborhood Y of H in G/H such that Y has a countable *cs*-network. Put $X = \pi^{-1}(Y)$. Then X is an open neighborhood of the neutral element e in G. By Lemma 3.1, X is separable. Let $B = \{b_m : m \in \mathbb{N}\}$ be a countable dense subset of X.

Since the subspace H of the topological group G is first-countable at the neutral element e of G, there is a countable family $\{U_n : n \in \mathbb{N}\}$ of open symmetric neighborhoods of e in G such that $U_{n+1}^3 \subset U_n \subset X$ for each $n \in \mathbb{N}$ and the family $\{U_n \cap H : n \in \mathbb{N}\}$ is a local base at e for H. Choose a countable *cs*-network $\{P_k : k \in \mathbb{N}\}$ for the $\aleph_0$-space Y.

Claim 1: X is an $\aleph_0$-space.

Put $\mathcal{F} = \{\pi^{-1}(P_k) \cap b_m U_n : k, m, n \in \mathbb{N}\}$. Then $\mathcal{F}$ is a countable family of subsets of X. Let $\{x_i\}_i$ be a sequence converging to a point x in X and U be a neighborhood of x in X. Then U is also a neighborhood of x in G. Take an open neighborhood V of e in G such that $xV^2 \subset U$. Since $\{U_n \cap H : n \in \mathbb{N}\}$ is a local base at e for H, there exists $n \in \mathbb{N}$ such that $U_n \cap H \subset V \cap H$. Since B is dense in X, and $xU_{n+1} \cap X$ is non-empty and open in X, then $b_m \in xU_{n+1}$ for some $m \in \mathbb{N}$. Since $\pi : G \to G/H$ is an open mapping [2, Theorem 1.5.1], then $\pi(xU_{n+1} \cap xV)$ is an open neighborhood of $\pi(x)$ in the space Y and the sequence $\{\pi(x_i)\}_i$ converges to $\pi(x)$ in Y, thus $\{\pi(x)\} \cup \{\pi(x_i) : i \geq i_0\} \subset P_k \subset \pi(xU_{n+1} \cap xV)$ for some $i_0, k \in \mathbb{N}$. We claim that $\pi^{-1}(P_k) \cap b_m U_{n+1} \subset U$.

Indeed, take any $z \in \pi^{-1}(P_k) \cap b_m U_{n+1}$. Then $\pi(z) \in P_k \subset \pi(xU_{n+1} \cap xV)$, thus $z \in (xU_{n+1} \cap xV)H = x(U_{n+1} \cap V)H$. Since $z \in b_m U_{n+1}$ and $b_m \in xU_{n+1}$, then $z \in xU_{n+1}^2$. Hence, $x^{-1}z \in [(U_{n+1} \cap V)H] \cap U_{n+1}^2$. There exist $a \in U_{n+1} \cap V, h \in H$ and $u \in U_{n+1}$ such that $x^{-1}z = ah = u^2$, then $h = a^{-1}u^2 \in U_{n+1}^3 \subset U_n$, so $x^{-1}z \in (U_{n+1} \cap V)(U_n \cap H)$. It follows that $z \in x(U_{n+1} \cap V)(U_n \cap H) \subset xV^2 \subset U$.

Further, it follows from $b_m \in xU_{n+1}$ that $x \in b_m U_{n+1}$, then there is $i_1 \geq i_0$ such that $x_i \in b_m U_{n+1}$ when $i \geq i_1$, thus $\{x\} \cup \{x_i : i \geq i_1\} \subset \pi^{-1}(P_k) \cap b_m U_{n+1}$. Hence, $\mathcal{F}$ is a countable *cs*-network for X, and this completes the proof of Claim 1.

By the homogeneity of the topological group G and Claim 1, G is also a local $\aleph_0$-space. Then G is a locally paracompact space, so G is a paracompact space by Lemma 3.2. Let $\mathcal{A}$ be an open cover of G by $\aleph_0$-subspaces. Since the property of being an $\aleph_0$-space is hereditary, we can assume that $\mathcal{A}$ is locally finite in G by the paracompactness of G. Since every point-countable family of open subsets in a separable space is countable, the family $\mathcal{A}$ is star-countable. It follows from Lemma 3.3 that $\mathcal{A} = \bigcup\{\mathcal{B}_\alpha : \alpha \in \Lambda\}$, where each subfamily $\mathcal{B}_\alpha$ is countable and $(\bigcup \mathcal{B}_\alpha) \cap (\bigcup \mathcal{B}_\beta) = \emptyset$ whenever $\alpha \neq \beta$. Put $X_\alpha = \bigcup \mathcal{B}_\alpha$ for each $\alpha \in \Lambda$. Then $G = \bigoplus_{\alpha \in \Lambda} X_\alpha$.

Claim 2: X_α is an $\aleph_0$-subspace for each $\alpha \in \Lambda$.

Put $\mathcal{B}_\alpha = \{B_{\alpha,n} : n \in \mathbb{N}\}$, where each $B_{\alpha,n}$ is an open $\aleph_0$-subspace of G. Put $\mathcal{P}_\alpha = \bigcup_{n\in\mathbb{N}} \mathcal{P}_{\alpha,n}$, where $\mathcal{P}_{\alpha,n}$ is a countable *cs*-network for the $\aleph_0$-space $B_{\alpha,n}$ for each $n \in \mathbb{N}$. It is easy to see that $\mathcal{P}_\alpha$ is a countable *cs*-network for X_α. Hence, X_α is an $\aleph_0$-space.

It follows from Claim 2 that G is a topological sum of $\aleph_0$-subspaces. □

Corollary 3.5. *Let H be a closed second-countable subgroup of a topological group G. If the quotient space G/H is an $\aleph_0$-space (resp., cosmic space), then G is also an $\aleph_0$-space (resp., cosmic space [2, Problems 4.6.C]).*

Remark 3.6. The condition "H is second-countable" is essential in Theorem 3.4 and Corollary 3.5, it cannot be replaced by the condition "H has a countable network", since there is a non-cosmic, Abelian topological group G_0 with a closed cosmic subgroup H_0 such that the quotient group G_0/H_0 is separable metric [27].

Next, we consider the spaces with a star-countable *cs*-network and similar networks. We recall some concepts related to networks for a topological space. Let $\mathcal{P}$ be a family of subsets of a topological space X. $\mathcal{P}$ is called a *k-network* [13] for X if whenever $K \subset U$ with K compact and U open in X, there exists a finite family $\mathcal{P}' \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{P}' \subset U$. $\mathcal{P}$ is called a *wcs*-network* [17] for X if, given a sequence $\{x_n\}_n$ converging to a point x in X and a neighborhood U of x in X, there exists a subsequence $\{x_{n_i}\}_i$ of the sequence $\{x_n\}_n$ such that $\{x_{n_i} : i \in \mathbb{N}\} \subset P \subset U$ for some $P \in \mathcal{P}$.

Every base is a k-network and a *cs*-network for a topological space, and every k-network or every *cs*-network is a *wcs**-network for a topological space, but the converse does not hold [16]. It is easy to verify that the following are equivalent for a space X: (1) X has a countable *cs*-network; (2) X has a countable k-network; (3) X has a countable *wcs**-network.

Lemma 3.7 *([15, Lemma 2.1.6]). Let $\mathcal{P}$ be a point-countable family of subsets of a space X. Then $\mathcal{P}$ is a k-network for X if and only if it is a wcs*-network for X and each compact subset of X is first-countable (or sequential).*

Theorem 3.8. *Let H be a closed second-countable subgroup of a topological group G. If the quotient space G/H has a star-countable cs-network (resp., wcs*-network, k-network), then G has also a star-countable cs-networks (resp., wcs*-network, k-network).*

Proof. Since the subspace H of the topological group G is first-countable at the neutral element e of G, there exists a countable family $\{U_n : n \in \mathbb{N}\}$ of open symmetric neighborhoods of e in G such that $U_{n+1}^3 \subset U_n$ for each $n \in \mathbb{N}$ and the family $\{U_n \cap H : n \in \mathbb{N}\}$ is a local base at e for H.

(1) Suppose that G/H has a star-countable *cs*-network (resp., *wcs**-network).

Let $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$ be a star-countable *cs*-network (resp., *wcs**-network) for the space G/H. For each $\alpha \in \Lambda$, the family $\{P_\alpha \cap P_\beta : \beta \in \Lambda\}$ is a countable *wcs**-network for P_α, thus P_α is a cosmic space, and P_α is separable. By Lemma 3.1, the set $\pi^{-1}(P_\alpha)$ is separable. Let $B_\alpha = \{b_{\alpha,m} : m \in \mathbb{N}\}$ be a countable dense subset of $\pi^{-1}(P_\alpha)$.

Put

$$\mathcal{F} = \{\pi^{-1}(P_\alpha) \cap b_{\alpha,m} U_n : \alpha \in \Lambda, \text{and } m, n \in \mathbb{N}\}.$$

Then $\mathcal{F}$ is a star-countable family of G. We shall show that $\mathcal{F}$ is a *cs*-network (resp., *wcs**-network) for G.

Let $\{x_i\}_i$ be a sequence converging to a point x in G and U be a neighborhood of x in G. Take an open neighborhood V at e in G such that $xV^2 \subset U$.

Case 1. $\mathcal{P}$ is a *cs*-network.

Since $\{U_n \cap H : n \in \mathbb{N}\}$ is a local base at e for H, there exists $n \in \mathbb{N}$ such that $U_n \cap H \subset V \cap H$. Since $\pi : G \to G/H$ is an open mapping, then $\{\pi(x)\} \cup \{\pi(x_i) : i \geq i_0\} \subset P_\alpha \subset \pi(xU_{n+1} \cap xV)$ for some $i_0 \in \mathbb{N}$ and some $\alpha \in \Lambda$. Since $x \in \pi^{-1}(P_\alpha)$, then $xU_{n+1} \cap \pi^{-1}(P_\alpha)$ is non-empty and open in the subspace $\pi^{-1}(P_\alpha)$. And since B_α is dense in $\pi^{-1}(P_\alpha)$, it follows that $b_{\alpha,m} \in xU_{n+1}$ for some $m \in \mathbb{N}$.

Claim: $\pi^{-1}(P_\alpha) \cap b_{\alpha,m}U_{n+1} \subset U$.

Indeed, take any $z \in \pi^{-1}(P_\alpha) \cap b_{\alpha,m}U_{n+1}$. Then $\pi(z) \in P_\alpha \subset \pi(xU_{n+1} \cap xV)$, thus $z \in x(U_{n+1} \cap V)H$. Since $z \in b_{\alpha,m}U_{n+1}$ and $b_{\alpha,m} \in xU_{n+1}$, then $z \in xU_{n+1}^2$. As in Claim 1 of the proof of Theorem 3.4, it follows that $z \in x(U_{n+1} \cap V)(U_n \cap H) \subset xV^2 \subset U$.

Further, it follows from $b_{\alpha,m} \in xU_{n+1}$ that $x \in b_{\alpha,m}U_{n+1}$, then there is $i_1 \geq i_0$ such that $x_i \in b_{\alpha,m}U_{n+1}$ whenever $i \geq i_1$, thus $\{x\} \cup \{x_i : i \geq i_1\} \subset \pi^{-1}(P_\alpha) \cap b_{\alpha,m}U_{n+1}$.

Hence, G has a star-countable *cs*-network.

Case 2. $\mathcal{P}$ is a *wcs**-network.

Since $\{U_n \cap H : n \in \mathbb{N}\}$ is a local base at e for H, there exists $n \in \mathbb{N}$ such that $U_{n+1} \cap H \subset V \cap H$. Since $\mathcal{P}$ is a *wcs**-network for G/H, there is a subsequence $\{\pi(x_{i_j})\}_j$ of the sequence $\{\pi(x_i)\}_i$ such that $\{\pi(x_{i_j}) : j \in \mathbb{N}\} \subset P_\alpha \subset \pi(xU_{n+1} \cap xV)$ for some $\alpha \in \Lambda$. We can assume that $x_{i_j} \in xU_{n+2}$ for each $j \in \mathbb{N}$ because the sequence $\{x_i\}_i$ converges x. Since $x_{i_1} \in \pi^{-1}(P_\alpha)$, then $x_{i_1}U_{n+2} \cap \pi^{-1}(P_\alpha)$ is non-empty and open in the subspace $\pi^{-1}(P_\alpha)$. And since B_α is dense in $\pi^{-1}(P_\alpha)$, it follows that $b_{\alpha,m} \in x_{i_1}U_{n+2} \subset xU_{n+2}^2$ for some $m \in \mathbb{N}$.

As in Case 1, we have the inclusion $\pi^{-1}(P_\alpha) \cap b_{\alpha,m}U_{n+1} \subset U$ (see Claim).

Hence, G has a star-countable *wcs**-network.

(2) Suppose that G/H has a star-countable k-network.

G has a star-countable *wcs**-network by Case 2. It follows from Lemma 3.7 that each compact subset of G/H is first-countable. Since the property "each compact subset is first-countable" is a three space property [2, Lemma 3.3.23 and Theorem 3.3.24], then each compact subset of G is first-countable, thus G has a star-countable k-network by Lemma 3.7. □

Question 3.9. *Let H be a closed second-countable subgroup of a topological group G. If the quotient space G/H has a point-countable (resp., compact-countable) cs-network, does G have a point-countable (resp., compact-countable) cs-networks?*

A partial answer to Question 3.9 will be given in Theorem 3.11.

Lemma 3.10 *([19, Theorem 3.6]). Let G be a sequential topological group with a point-countable k-network. Then G is a metrizable space or a topological sum of cosmic spaces.*

Theorem 3.11. *Suppose that H is a closed, second-countable and invariant subgroup of a topological group G. If the quotient group G/H is a sequential space with a point-countable cs-network (resp., k-network, wcs*-network), then G has a point-countable cs-network (resp., k-network, wcs*-network).*

Proof. Since the space G/H is sequential, it has a point-countable k-network by Lemma 3.7. It follows from Lemma 3.10 that the quotient group G/H is a metrizable space or a topological sum of cosmic spaces. If G/H is metrizable, then the group G is metrizable [2, Corollary 1.5.21]. We can assume that G/H is a topological sum of cosmic spaces. Then G/H has a point-countable *cs*-network (resp., k-network, *wcs**-network) $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$ such that P_α is a cosmic subset for each $\alpha \in \Lambda$. Since each $\pi^{-1}(P_\alpha)$ is separable by Lemma 3.1, it can be shown that the space G has a point-countable *cs*-network (resp., k-network, *wcs**-network) by a method similar to the one in the proof of Theorem 3.8. □

4. Quotients with respect to locally compact metrizable subgroups

There exists a closed second-countable and invariant subgroup H of a topological group G such that the quotient group G/H is compact, but G is not a k-space.

Example 4.1 *([1, Theorem 2.19]).* ($2^{\omega_1} = 2^{\omega}$) There exists an Abelian topological group G with a closed second-countable subgroup H such that the quotient group $G/H(= \{0,1\}^{\omega_1})$ is compact, and G has a G_δ-diagonal, but G is not a p-space.

In the above Arhangel'skiĭ's example, the group G is not a k-space. In fact, if G is a k-space, then G is also a sequential space [20, Theorem 7.3]. Since sequentiality is preserved by a quotient mapping [9], then the quotient group G/H is a sequential space, a contradiction.

Arhangel'skiĭ [1] established some extension theorems in topological groups in which he considered quotients with respect to locally compact subgroups or locally compact metrizable subgroups. The following facts are obtained in [1,3]. Let G be a topological group and H a subgroup of G.

(a) If the quotient space G/H is a Čech-complete space (resp., Σ-space, strong Σ-space, p-space, k-space, paracompact space), then so is the group G provided H is locally compact.
(b) If the quotient space G/H is a Fréchet space (resp., strongly Fréchet space, or has countable tightness), then so is the group G provided H is locally compact and metrizable.

In this section we continue to consider the extensions of topological groups with respect to locally compact metrizable subgroups. The following lemma about quotients with respect to locally compact subgroups was proved by Arhangel'skiĭ [1].

Lemma 4.2 *([2, Theorem 3.2.2]). Suppose that G is a topological group, H is a locally compact subgroup of G, and $\pi : G \to G/H$ is the natural quotient mapping of G onto the quotient space G/H. Then there exists an open neighborhood U of the neutral element e such that $\pi(\overline{U})$ is closed in G/H and the restriction $\pi|_{\overline{U}} : \overline{U} \to \pi(\overline{U})$ is a perfect mapping, thus π is an open locally perfect mapping.*

Theorem 4.3. *Suppose that H is a locally compact metrizable subgroup of a topological group G. If the quotient space G/H is sequential, then G is also sequential.*

Proof. By Lemma 4.2, there exists an open neighborhood U of the neutral element e in G such that $\pi|_{\overline{U}} : \overline{U} \to \pi(\overline{U})$ is a perfect mapping and $\pi(\overline{U})$ is closed in G/H.

Claim 1: Suppose that $\{x_n\}_n$ is a sequence in $\overline{U}$ such that $\{\pi(x_n)\}_n$ is a convergent sequence in $\pi(\overline{U})$. If x is an accumulation point of the sequence $\{x_n\}_n$, there exists a subsequence of $\{x_n\}_n$ which converges to x.

Since $\pi|_{\overline{U}}$ is perfect, each subsequence of the sequence $\{x_n\}_n$ has an accumulation point in $\overline{U}$. Set $F = \pi^{-1}(\pi(x)) \cap \overline{U}$. Since $\pi^{-1}(\pi(x)) = xH$ is metrizable and G is regular, there is a sequence $\{U_k\}_k$ of open subsets in G such that $\overline{U_{k+1}} \subset U_k$ for each $k \in \mathbb{N}$ and $\{x\} = F \cap \bigcap_{k\in\mathbb{N}} U_k$. Take a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ such that $x_{n_k} \in U_k$ for each $k \in \mathbb{N}$. Let p be an accumulation point of a subsequence of the sequence $\{x_{n_k}\}_k$. Then $\pi(p) = \pi(x)$ and $p \in \bigcap_{k\in\mathbb{N}} \overline{U_k}$, thus $p = x$. It follows that x is the unique accumulation point of every subsequence of $\{x_{n_k}\}_k$, then $x_{n_k} \to x$.

Take an open neighborhood V of e in G such that $\overline{V} \subset U$.

Claim 2: If C is sequentially closed in $\overline{V}$, then $\pi(C)$ is closed in $\pi(\overline{V})$.

Let $\{y_n\}_n$ be a sequence in $\pi(C)$ such that $y_n \to y$ in $\pi(\overline{V})$. We shall show that $y \in \pi(C)$. Take $x_n \in C$ with $\pi(x_n) = y_n$ for each $n \in \mathbb{N}$. Since each subsequence of the sequence $\{x_n\}_n$ has an accumulation point, by Claim 1, there are a point $x \in \pi^{-1}(y)$ and a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ such that $x_{n_k} \to x$. Since C is sequentially closed, then $x \in C$, and $y \in \pi(C)$. This shows that $\pi(C)$ is sequentially closed in $\pi(\overline{V})$. Since $\pi|_{\overline{U}} : \overline{U} \to \pi(\overline{U})$ is a closed mapping and $\pi(\overline{U})$ is closed in G/H, $\pi(\overline{V})$ is closed in G/H. Since G/H is sequential, then $\pi(\overline{V})$ is also sequential, so $\pi(C)$ is closed in $\pi(\overline{V})$.

Claim 3: $\overline{V}$ is a sequential subspace.

Suppose that there exists a non-closed, sequentially closed subset A of $\overline{V}$. Take a point $x \in \mathrm{cl}_{\overline{V}}(A) \setminus A$. Clearly, $\mathrm{cl}_{\overline{V}}(A) = \overline{A}$. Let $f = \pi|_{\overline{V}} : \overline{V} \to \pi(\overline{V})$. The set $B = A \cap f^{-1}(f(x))$ is sequentially closed as a closed subset of A and, since the fiber $f^{-1}(f(x)) = (\pi^{-1}\pi(x)) \cap \overline{V}$ is sequential, B is closed in $\overline{V}$. Since $x \notin B$, there is an open neighborhood W of x in $\overline{V}$ such that $\overline{W} \cap B = \emptyset$. Then $C = \overline{W} \cap A$ is also sequentially closed as a closed subset of A and $x \in \overline{C} \setminus C$. It follows that $C \cap f^{-1}(f(x)) = \overline{W} \cap B = \emptyset$, then $f(x) \in \overline{f(C)} \setminus f(C)$, so $f(C) = \pi(C)$ is not closed in $\pi(\overline{V})$, a contradiction with Claim 2.

By Claim 3 and the homogeneity of G, G is a locally sequential space. Hence, G is a sequential space. □

Remark 4.4. There exist two Fréchet topological groups G and H such that the product space $G \times H$ is not of countable tightness (see [22, Theorem 6.6] or [26]). Put $H' = \{e\} \times H$, where e is the neutral element in G. Then H' is a closed invariant subgroup of $G \times H$, and the quotient group $(G \times H)/H'$ is isomorphic to G. Thus, H' and $(G \times H)/H'$ is sequential, but $G \times H$ is not sequential. Therefore, sequentiality is not a three space property.

In view of Theorems 3.11 and 4.3 the following corollary is immediate.

Corollary 4.5. *Suppose that H is a locally compact, second-countable and invariant subgroup of a topological group G. If the quotient group G/H is a sequential space with a point-countable cs-network (resp., k-network, wcs*-network), then G is also a sequential space with a point-countable cs-network (resp., k-network, wcs*-network).*

Theorem 4.6. *Suppose that H is a locally compact metrizable subgroup of a topological group G. If the quotient space G/H is strictly Fréchet, then G is also strictly Fréchet.*

Proof. By Lemma 4.2, there exists an open neighborhood U of the neutral element e in the topological group G such that $\pi|_{\overline{U}} : \overline{U} \to \pi(\overline{U})$ is a perfect mapping and $\pi(\overline{U})$ is closed in G/H. Put $f = \pi|_{\overline{U}} : \overline{U} \to \pi(\overline{U})$.

It is obvious that $f(\overline{U}) = \pi(\overline{U})$ is strictly Fréchet, and $f^{-1}(f(b)) = \pi^{-1}(\pi(b)) \cap \overline{U} = bH \cap \overline{U}$ is compact and metrizable for each $b \in \overline{U}$. By Lemma 2.10, $\overline{U}$ is strictly Fréchet. Thus, G is locally strictly Fréchet, and G is strictly Fréchet. □

A subset A of a topological space X is called *sequentially open* [9] if $X \setminus A$ is sequentially closed. A space X is called *sequentially connected* [8] if X cannot be represented as the union of two non-empty disjoint and sequentially open subsets of X. Every sequentially connected space is connected, and every connected and sequential space is sequentially connected. It is known that connectedness is a three space property [2, Exercises 1.5.e].

Theorem 4.7. *Suppose that H is a locally compact metrizable connected and invariant subgroup of a topological group G. If the quotient group G/H is sequentially connected, then G is also sequentially connected.*

Proof. If the topological group G is not sequentially connected, there are two non-empty, disjoint and sequentially open subsets A and B of G such that $G = A \cup B$. If $y \in G/H$, then $\pi(x) = y$ for some $x \in G$, so $\pi^{-1}(y) = xH$ is sequentially connected, thus $\pi^{-1}(y) \subset A$ or $\pi^{-1}(y) \subset B$. It follows that there exist two non-empty disjoint subsets C and D of G/H such that $G/H = C \cup D$, $\pi^{-1}(C) = A$ and $\pi^{-1}(D) = B$.

Next, we shall show that C, D are sequentially open in G/H. This gives a contradiction since G/H is sequentially connected.

If C is not sequentially open in G/H, there is a sequence $\{y_n\}_n$ in G/H such that $y_n \to y \in C$ with $y_n \notin C$ for each $n \in \mathbb{N}$. Then $y_n y^{-1} \to e' \in Cy^{-1}$ in G/H, where e' is the neutral element in G/H. Let U be the open neighborhood of e in G as the proof of Theorem 4.3, where e is the neutral element in G. Since $\pi(U)$ is open, we can assume that $y_n y^{-1} \in \pi(U)$ for each $n \in \mathbb{N}$. By Claim 1 in the proof of Theorem 4.3, there exists a convergent sequence $\{x_k\}_k$ in U such that $x_k \to x$ for some $x \in G$ and $\pi(x_k) = y_{n_k} y^{-1}$ for each $k \in \mathbb{N}$, here $\{y_{n_k}\}_k$ is a subsequence of $\{y_n\}_n$ with $n_k \to \infty$. Then $\pi(x) = e' \in Cy^{-1}$, so $x \in \pi^{-1}(Cy^{-1}) = \pi^{-1}(C)\pi^{-1}(y^{-1})$. Since $\pi^{-1}(C) = A$ is sequentially open in G, $\pi^{-1}(C)\pi^{-1}(y^{-1})$ is also sequentially open in G, then $x_k \in \pi^{-1}(C)\pi^{-1}(y^{-1})$ for some $k \in \mathbb{N}$, so $y_{n_k} y^{-1} = \pi(x_k) \in Cy^{-1}$, and $y_{n_k} \in C$, a contradiction. Hence, C is sequentially open. By the same reason, D is also sequentially open. □

Question 4.8. *Suppose that H is a locally compact and sequentially connected subgroup of a topological group G. Is G sequentially connected if the quotient space G/H is sequentially connected?*

Addendum

The answer to Question 4.8 is "yes". The following result is obtained in [18, Theorem 3.5]: Let H be a closed, sequentially connected, feathered subgroup of a Hausdorff topological group G. If the quotient space G/H is sequentially connected, then so is G.

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