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SUBMETRIZABILITY IN PARATOPOLOGICAL GROUPS

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Abstract. In this paper the following question posed by Mikhail Tkachenko in *Paratopological and semitopological groups vs topological groups* [to appear in Recent Progress in General Topology III] is considered: Does a Hausdorff or regular paratopological group G with $l(G) \cdot \psi(G) \leq \omega$ admit a continuous bijection onto a Hausdorff space with a countable base? Some conditions under which G admits a weaker metrizable topological group topology are given. It is shown that every Hausdorff 2-oscillating paratopological group G with $Hs(G) \cdot \psi(G) \leq \omega$ is submetrizable. If, in addition, G is ω-balanced, then G admits a weaker metrizable topological group topology.

1. Introduction

If multiplication in a group is jointly continuous, then this object is called a *paratopological group*. If, in addition, the inversion in the group is continuous, then it is called a *topological group*.

A space X is called *submetrizable* if there exists a continuous bijection X onto a metrizable space. It is well known that every topological group in which every point is a G_δ-set is submetrizable [2, Theorem 3.3.16]. This motivated Alexander Arhangel'skii and Mikhail Tkachenko to pose the following question.

Question 1.1 ([2, Open problem 3.3.1]). Suppose that G is a Hausdorff (regular) paratopological group in which every point is a G_δ-set. Is G submetrizable?

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Following [4], a paratopological group G that has a weaker Hausdorff topological group topology will be called *subtopological*. Recently, Manuel Fernández [4] posed the following question.

Question 1.2 ([4, Question 3.13])**.** Does every Hausdorff first-countable subtopological group admit a weaker Hausdorff first-countable topological group topology?

It is well known that every topological group G is first-countable if and only if it is metrizable. Therefore, we can reformulate Question 1.2 by asking whether every Hausdorff first-countable subtopological group admits a weaker metrizable topological group topology. Recently, Tkachenko [14] posed the following question.

Question 1.3 ([14])**.** Does a Hausdorff or regular paratopological group G with $l(G) \cdot \psi(G) \leq \omega$ admit a continuous bijection onto a Hausdorff space with a countable base?

Fucai Lin and Chuan Liu [7, Example 3.3] gave a negative answer to Question 1.1 for Hausdorff paratopological groups and they also discussed what restrictions on a Hausdorff first-countable paratopological group G ensure that G is submetrizable. For Question 1.2, Fernández [4] proved the following result.

Theorem 1.4 ([4, Proposition 3.11])**.** *Any Hausdorff first-countable 3-oscillating paratopological group admits a weaker metrizable topological group topology.*

As for Question 1.3, Lin and Liu established the following theorem.

Theorem 1.5 ([7, Theorem 3.6])**.** *Every regular ω-narrow first-countable paratopological group admits a continuous bijection onto a Hausdorff space with a countable base.*

After the above discussion, the question of finding some topological properties which imply that a paratopological group with countable pseudocharacter admits a weaker metrizable topological group topology (or admits a continuous bijection onto a Hausdorff space with a countable base) arises in a natural way. In this framework, our results generalize Theorem 1.5 and other results in [7], and we also give some partial answers to questions 1.2 and 1.3. We mainly show that every Hausdorff ω-narrow and ω-balanced paratopological group G with $Hs(G) \cdot \psi(G) \leq \omega$ admits a continuous bijection onto a Hausdorff space with a countable base (Theorem 2.3), and that every Hausdorff 2-oscillating paratopological group G with $Hs(G) \cdot \psi(G) \leq \omega$ is submetrizable (Theorem 3.5). We also establish that every feebly compact paratopological group G, such that the identity is a regular G_δ-set, admits a weaker metrizable topological group topology (Theorem 2.7).

All spaces in this paper satisfy the T_0 separation axiom. $l(X)$, $\chi(X)$, and $\psi(X)$ denote the Lindelöf number, character, and pseudocharacter of a space X, respectively.

2. ω-Narrow and ω-Balanced Paratopological Groups

First, we give some partial answers to Question 1.3 in this section. Recall that a paratopological group G is *ω-narrow* [2, p. 117] if, for every neighborhood U of the identity in G, there exists a countable set $A \subseteq G$ such that $AU = G = UA$. Also G is called *ω-balanced* [2, p. 164] if, for every neighborhood U of identity e in G, there exists a family γ of open neighborhoods of e in G with $|\gamma| \leq \omega$ such that, for each $x \in G$, one can find $V \in \gamma$ satisfying $xVx^{-1} \subseteq U$.

For a Hausdorff paratopological group G with identity e, the *Hausdorff number* of G [13], denoted by $Hs(G)$, is the minimum cardinal number κ such that, for every neighborhood U of e in G, there exists a family γ of neighborhoods of e such that $\bigcap_{V\in\gamma} VV^{-1} \subseteq U$ and $|\gamma| \leq \kappa$.

Remark 2.1. Clearly, every Hausdorff topological group G has $Hs(G) = 1$, and every first-countable (or Lindelöf) Hausdorff paratopological group G has $Hs(G) \leq \omega$ [13].

A subset U of a space X is called *regular open* if $U = \mathrm{Int}(\overline{U})$. Similarly, a subset F of a space X is called *regular closed* if $F = \overline{\mathrm{Int}(F)}$. Given a space (X, τ), denote by τ' the topology on X whose base consists of regular open subsets of (X, τ). The space (X, τ') is said to be the *semiregularization* of (X, τ) and is denoted by X_{sr}. It is easy to see that $\tau' \subset \tau$ and that the spaces (X, τ) and (X, τ') have the same regular open and regular closed subsets.

The operation of semiregularization was defined by M. H. Stone in [12] and studied by Miroslav Katetov in [6]. The following proposition shows that "regular" can be weakened to "Hausdorff" in Theorem 1.5.

Proposition 2.2. *Every Hausdorff ω-narrow first-countable paratopological group admits a continuous bijection onto a Hausdorff space with a countable base.*

Proof. Let G be a Hausdorff ω-narrow first-countable paratopological group. According to Theorem 1.5, it suffices to show that G admits a continuous bijection onto a regular ω-narrow first-countable paratopological group. Indeed, let G_{sr} be the semiregularization of G. Since G is a Hausdorff paratopological group, it follows from [10, Example 1.9] that G_{sr} is a regular paratopological group. One can easily verify that G_{sr} is

ω-narrow and first-countable. Thus, the identity map $i : G \to G_{sr}$ is a continuous bijection. $\square$

Theorem 2.3. *Every Hausdorff ω-narrow and ω-balanced paratopological group G with $Hs(G) \cdot \psi(G) \leq \omega$ admits a continuous bijection onto a Hausdorff space with a countable base.*

Proof. According to Proposition 2.2, it suffices to prove that G admits a continuous isomorphism onto a Hausdorff ω-narrow first-countable paratopological group.

Suppose that $\{e\} = \bigcap_{n\in\omega} U_n$, where U_n is an open set in G for each $n \in \omega$ and e is the identity in G. Since G is ω-balanced with $Hs(G) \leq \omega$, it follows from [13, Theorem 2.7] (see also [15, Lemma 2.3]) that there exists a continuous homomorphism π_{U_n} of G onto a Hausdorff first-countable paratopological group H_{U_n} such that $\pi_{U_n}^{-1}(V_n) \subseteq U_n$ for some open neighborhood V_n of the identity in H_{U_n} for each $n \in \omega$. Put $H = \prod_{n\in\omega} H_{U_n}$ and define $\pi = \Delta_{n\in\omega}\pi_{U_n}$ as the diagonal product of the family $\{\pi_{U_n} | n \in \omega\}$. It is obvious that π is a continuous isomorphism. Then $\pi(G)$ is a Hausdorff first-countable paratopological group, since $\pi(G) \subseteq \prod_{n\in\omega} H_{U_n}$ is Hausdorff and first-countable. It remains to show that $\pi(G)$ is ω-narrow. This follows from the fact that $\pi(G)$ is a continuous homomorphic image of the ω-narrow paratopological group G. $\square$

Remark 2.4. It is obvious that every regular ω-narrow and first-countable paratopological group G is ω-balanced and $Hs(G)\cdot\psi(G) \leq \omega$, while there is an ω-balanced paratopological group H such that $Hs(H)\cdot\psi(H) \leq \omega$ and $\chi(H) > \omega$, so Theorem 2.3 generalizes Theorem 1.5. Indeed, there exists a topological group G such that $Hs(G)\cdot\psi(G) \leq \omega$ and $\chi(G) > \omega$. For example, one can take a completely regular non-discrete topological space X with a countable network, then the free Abelian topological group $A(X)$ is regular and has a countable network, but $\chi(A(X)) > \omega$ according to [2, Corollary 7.1.17 and Theorem 7.1.20]. By Remark 2.1, $A(X)$ is an ω-narrow and ω-balanced topological group with $Hs(A(X))\cdot\psi(A(X)) \leq \omega$.

For a paratopological group G with topology τ, one defines the *conjugate topology* τ^{-1} on G by $\tau^{-1} = \{U^{-1} | U \in \tau\}$. Then $G' = (G, \tau^{-1})$ is also a paratopological group, and the inversion $x \to x^{-1}$ is a homeomorphism of G onto G'. The upper bound $\tau^* = \tau \vee \tau^{-1}$ is a topological group topology on G, and we call $G^* = (G, \tau^*)$ the topological group *associated to* G. A paratopological group G is called *totally ω-narrow* [11] if G^* is ω-narrow.

Clearly, every totally ω-narrow paratopological group G is ω-narrow. And it is well known that every totally ω-narrow paratopological group G is ω-balanced [11]; thus, the following result is obvious by Theorem 2.3.

Corollary 2.5. *Every Hausdorff totally ω-narrow paratopological group G with $Hs(G)\cdot\psi(G)\leq\omega$ admits a continuous bijection onto a Hausdorff space with a countable base.*

The following corollary follows directly from Remark 2.1 and Theorem 2.3. It gives a partial answer to Question 1.3.

Corollary 2.6. *Every Hausdorff ω-balanced paratopological group G with $l(G)\cdot\psi(G)\leq\omega$ admits a continuous bijection onto a Hausdorff space with a countable base.*

Recall that a space X is called *feebly compact* if every locally finite family of open sets in X is finite. A subset $A\subseteq X$ is called a *regular G_δ-set* if there exists a countable family $\{U_n : n\in\omega\}$ of open sets in X such that $A=\bigcap_{n\in\omega}U_n=\bigcap_{n\in\omega}\overline{U_n}$.

Theorem 2.7. *Let G be a feebly compact paratopological group in which the identity e is a regular G_δ-set. Then G admits a weaker metrizable topological group topology.*

Proof. Let G_{sr} be the semiregularization of G. Then, from [10, Example 1.9], it follows that G_{sr} is a T_3 paratopological group topology. Since the set $\{e\}$ is a regular G_δ-set in G, one can easily verify that the set $\{e\}$ is a G_δ-set in G_{sr}. Hence, it is obvious that G_{sr} satisfies the T_1 separation axiom. Thus, G_{sr} is regular. Since every regular feebly compact paratopological group is a topological group [2, Theorem 2.4.1], G_{sr} is a completely regular feebly compact topological group with $\psi(G_{sr})\leq\omega$. It is well known that every completely regular feebly compact space with countable pseudocharacter is first-countable [8], so we obtain that G_{sr} is a metrizable topological group. This completes the proof. $\square$

Corollary 2.8. *Every feebly compact Hausdorff paratopological group G with $\psi(G)\cdot Hs(G)\leq\omega$ admits a weaker metrizable topological group topology.*

Proof. According to Theorem 2.7, it is enough to prove that the set $\{e\}$ is a regular G_δ-set in G, where e is the identity of G. Suppose that the family $\{U_n : n\in\omega\}$ of open neighborhoods at e is such that $\{e\}=\bigcap_n U_n$. Since $Hs(G)\leq\omega$, there exists a countable family γ_n of open neighborhoods at e such that $\bigcap_{V\in\gamma_n}VV^{-1}\subseteq U_n$ for each U_n. Put $\gamma=\bigcup_{n\in\omega}\gamma_n$. One can easily verify that $\{e\}=\bigcap_{V\in\gamma}V\subseteq\bigcap_{V\in\gamma}\overline{V}\subseteq\bigcap_{V\in\gamma}VV^{-1}\subseteq\bigcap_{n\in\omega}U_n=\{e\}$. This completes the proof. $\square$

Theorem 2.9. *Every feebly compact Hausdorff paratopological group G of countable π-character admits a weaker metrizable topological group topology.*

Proof. Let G_{sr} be the semiregularization of G. Then from [10, Example 1.9] it follows that G_{sr} is a regular paratopological group. Since every regular feebly compact paratopological group is a topological group [2, Theorem 2.4.1], G_{sr} is a completely regular feebly compact topological group. From the fact that G has countable π-character, it follows that so does G_{sr}. Indeed, let $\mathcal{C}$ be a countable π-base at the identity in G. Then one can easily verify that $\mathcal{C}' = \{\mathrm{Int}(\overline{V})|V \in \mathcal{C}\}$ is a countable π-base at the identity in G_{sr}. It is well known that every topological group with a countable π-character is first-countable, so G_{sr} is a metrizable topological group. $\square$

Corollary 2.10 ([7, Theorem 3.14])**.** *If G is a Hausdorff feebly compact paratopological group with $\chi(G) \leq \omega$, then G is submetrizable.*

3. 2-Oscillating Paratopological Groups

In this section we give some conditions under which a paratopological group G with $\psi(G) \leq \omega$ admits a weaker metrizable topological group topology. Following [3], a paratopological group G is called *2-oscillating* (*3-oscillating*) provided that, for every open neighborhood U of the identity e in G, there is an open neighborhood V of e such that $V^{-1}V \subseteq UU^{-1}$ ($V^{-1}VV^{-1} \subseteq UU^{-1}U$). Clearly, 2-oscillating paratopological groups are 3-oscillating. For 2-oscillating paratopological groups, we have a more general result in Theorem 3.5 than in Theorem 1.4. Some auxiliary facts must be established before we present the proof of Theorem 3.5. Lemmas 3.1 and 3.2 are obvious.

Lemma 3.1. (1) *Every subgroup of a 2-oscillating paratopological group is 2-oscillating.*

(2) *The topological product of arbitrarily many 2-oscillating paratopological groups is 2-oscillating.*

Following [3], under the *2-oscillator topology* on a paratopological group G, we understand the topology τ_2, consisting of the sets $U \subseteq G$ such that, for each $x \in U$, there is an open neighborhood V of the identity in G such that with $x(VV^{-1}) \subseteq U$. It is clear that τ_2 is weaker than the original topology of G.

Lemma 3.2. *Let (G, τ) be a Hausdorff paratopological group. Then $\chi(G, \tau_2) \leq \chi(G, \tau)$ and $\psi(G, \tau_2) \leq Hs(G, \tau) \cdot \psi(G, \tau)$, where τ_2 is the 2-oscillator topology on the paratopological group (G, τ).*

Lemma 3.3. *Let $\mathcal{N}(e)$ be the family of open neighborhoods of the identity e in a paratopological group G. Suppose that a subfamily $\gamma \subseteq \mathcal{N}(e)$ satisfies the following conditions:*

(a) *for each $U \in \gamma$, there exists $V \in \gamma$ such that $V^2 \subseteq U$;*
(b) *for every $U \in \gamma$ and every $a \in G \setminus U$, there exists $V \in \gamma$ such that $a \notin VV^{-1}$;*
(c) *for every $U \in \gamma$ and every $a \in G$, there exists $V \in \gamma$ such that $aVa^{-1} \subseteq U$.*

Then the set $H = \bigcap \gamma$ is a closed invariant subgroup of G.

Proof. Firstly, we shall show $H = \bigcap_{V\in\gamma} VV^{-1}$. The inclusion $H \subseteq \bigcap_{V\in\gamma} VV^{-1}$ is obvious. Take any $x \notin H$. Then there exist $U, V \in \gamma$ such that $x \notin U$ and $x \notin VV^{-1}$ according to (b), so $\bigcap_{V\in\gamma} VV^{-1} \subseteq H$, which implies $H = \bigcap_{V\in\gamma} VV^{-1}$. In fact, we have proved that H is a closed set, since for each $x \notin H$, there exists $V \in \gamma$ such that $x \notin VV^{-1}$, so $\emptyset = xV \cap V$ and $\emptyset = xV \cap H$ for $H \subseteq V$.

Now we shall show that H is an invariant subgroup of G. Take any $x, y \in H$ and $U \in \gamma$. Then there exists $V \in \gamma$ such that $V^2 \subseteq U$ according to (a), so $xy \in VV \subseteq U$, which implies $HH \subseteq H$. So we have $HH = H$. We also have $H^{-1} = H$, since $H^{-1} = (\bigcap_{V\in\gamma} VV^{-1})^{-1} = \bigcap_{V\in\gamma} (VV^{-1})^{-1} = \bigcap_{V\in\gamma} VV^{-1} = H$. Therefore, H is a subgroup of G. For each $a \in G$, we have $aHa^{-1} = a(\bigcap_{V\in\gamma} V)a^{-1} = \bigcap_{V\in\gamma}(aVa^{-1}) \subseteq \bigcap_{V\in\gamma} V = H$ by (c), which implies that H is an invariant subgroup of G. □

A neighborhood V of the identity e in a paratopological group G is called *ω-good* [11] if there exists a countable family γ of open neighborhoods of e in G such that, given any $x \in V$, we can find $W \in \gamma$ with $xW \subseteq V$. It is immediate from the definition that the intersection of finitely many ω-good sets is ω-good. In [11], it proved that every paratopological group G has a local base at the identity consisting of ω-good sets.

Lemma 3.4. *Let G be an ω-balanced 2-oscillating paratopological group with $Hs(G) \leq \omega$. Then for every open neighborhood U of the identity in G, there exists a continuous homomorphism π of G onto a Hausdorff first-countable 2-oscillating paratopological group H such that $\pi^{-1}(V) \subseteq U$ for some open neighborhood V of the identity in H.*

Proof. Take any open neighborhood U of identity e in G. Let $\mathcal{N}(e)$ be the family of all open neighborhoods of e in G. Denote by $\mathcal{N}^*(e)$ the subfamily of $\mathcal{N}(e)$ consisting of all ω-good sets. It follows from [11, Lemma 2.5] that $\mathcal{N}^*(e)$ is a local base for G at e.

Choose $U_0^* \in \mathcal{N}^*(e)$ satisfying $U_0^* \subseteq U$. Put $\gamma_0 = \{U_0^*\}$. Suppose that for some $n \in \omega$ we have defined families $\gamma_0, \cdots, \gamma_n$ satisfying the following conditions for each $k \leq n$:

(a) $\gamma_k \subseteq \mathcal{N}^*(e)$ and $|\gamma_k| \leq \omega$;
(b) $\gamma_k \subseteq \gamma_{k+1}$;
(c) γ_k is closed under finite intersections;
(d) for every $U \in \gamma_k$, there exists $V \in \gamma_{k+1}$ such that $V^2 \subseteq U$;
(e) for each $x \in G$ and $U \in \gamma_k$, there exists $V \in \gamma_{k+1}$ such that $xVx^{-1} \subseteq U$;
(f) $\bigcap_{V \in \gamma_{k+1}} VV^{-1} \subseteq U$, for each $U \in \gamma_k$;
(g) for each $U \in \gamma_k$, there exists $V \in \gamma_{k+1}$ such that $V^{-1}V \subseteq UU^{-1}$.

Clearly, we assume that $k+1 \leq n$ in (b) and (d)–(g). Since γ_n is countable, we can find a countable family $\lambda_{n,1} \subseteq \mathcal{N}^*(e)$ such that each $U \in \gamma_n$ contains the square of some element $V \in \lambda_{n,1}$. Since the group G is ω-balanced, there exists a countable family $\lambda_{n,2} \subseteq \mathcal{N}^*(e)$ such that for each $x \in G$ and $U \in \lambda_{n,2}$, there exists $V \in \gamma_{k+1}$ such that $xVx^{-1} \subseteq U$. Further, we use the condition $Hs(G) \leq \omega$ to find a countable family $\lambda_{n,3} \subseteq \mathcal{N}^*(e)$ such that $\bigcap_{V \in \lambda_{n,3}} VV^{-1} \subseteq U$, for each $U \in \gamma_n$. Finally, since G is 2-oscillating, we can find a countable family $\lambda_{n,4} \subseteq \mathcal{N}^*(e)$ such that for each $U \in \gamma_n$, there exists $V \in \lambda_{n,4}$ such that $V^{-1}V \subseteq UU^{-1}$. Let γ_{n+1} be the minimal family containing $\gamma_n \cup \bigcup_{i=1}^4 \lambda_{n,i}$ and closed under finite intersections. It is clear that γ_{n+1} is countable and that the families $\gamma_0, \cdots, \gamma_{n+1}$ satisfy (a)–(g).

It is easy to see that the family $\gamma = \bigcup_{i \in \omega} \gamma_i$ is countable and satisfies conditions (a)–(c) of Lemma 3.3. Therefore, $N = \bigcap \gamma$ is a closed invariant subgroup of G. Let $p : G \to G/N$ be the canonical homomorphism. Clearly, γ satisfies conditions (i)–(vi) of [14, Theorem 2.7]. Hence, according to the proof of [14, Theorem 2.7], we obtain that the family $\mu = \{p(V) | V \in \gamma\}$ is a local base at the identity of $H = G/N$ for a Hausdorff paratopological group topology on H. Thus, it remains to show that H is 2-oscillating. This follows directly from the fact that γ satisfies (g). □

Theorem 3.5. *Every 2-oscillating Hausdorff paratopological group G with $Hs(G) \cdot \psi(G) \leq \omega$ is submetrizable. If, in addition, G is ω-balanced, then G admits a weaker metrizable topological group topology.*

Proof. Since G is a Hausdorff 2-oscillating paratopological group, (G, τ_2) is a topological group satisfying the T_1 separation axiom [3], where τ_2 is the 2-oscillator topology on the paratopological group G. Since $Hs(G) \cdot \psi(G) \leq \omega$ holds, according to Lemma 3.2 we have $\psi(G, \tau_2) \leq \omega$. From

[2, Theorem 3.3.16] it follows that (G, τ_2) is submetrizable, which implies that G is submetrizable as well.

Now suppose that G is ω-balanced. According to Theorem 1.4, it is enough to prove that G admits a continuous isomorphism onto a first-countable Hausdorff 2-oscillating paratopological group H. Suppose that $\{e\} = \bigcap_{n\in\omega} U_n$, where U_n is an open neighborhood at identity e of G for each $n \in \omega$. From Lemma 3.4, it follows that there exists a continuous homomorphism π_n of G onto a first-countable Hausdorff 2-oscillating paratopological group H_n such that $\pi_n^{-1}(V) \subseteq U_n$ for some open neighborhood V of the identity in H_n for each $n \in \omega$. Define $\pi = \Delta_n \pi_n : G \to \prod_{n\in\omega} H_n$ as the diagonal product of the family $\{\pi_n | n \in \omega\}$. Clearly, π is a continuous isomorphism. From Lemma 3.1, it follows that the $\pi(G)$ is a first-countable Hausdorff 2-oscillating paratopological group. This completes the proof. □

Remark 3.6. Every first-countable Hausdorff paratopological group G is ω-balanced and satisfies $Hs(G) \cdot \psi(G) \leq \omega$. However, there exists a paratopological group G such that $Hs(G) \cdot \psi(G) \leq \omega$ and $\chi(G) > \omega$ according to Remark 2.4. We don't know whether Theorem 3.5 is true for 3-oscillating paratopological groups. Indeed, Lemma 3.4 is true for 3-oscillating paratopological groups; however, we don't know whether Lemma 3.2 is true for 3-oscillating paratopological groups.

As an application of Theorem 3.5, we have the following corollary, which gives a partial answer to Question 1.3. We recall that a paratopological group G is *saturated* [5] if, for any neighborhood U of the identity in G, the set U^{-1} has a nonempty interior in G. It is well known that the class of 2-oscillating paratopological groups contains all saturated paratopological groups [3, Proposition 3].

Corollary 3.7. *Every Hausdorff Baire paratopological group G with $l(G) \cdot \psi(G) \leq \omega$ admits a continuous bijection onto a separable metrizable space.*

Proof. Since G is Lindelöf and Baire, G is saturated by [1, Theorem 2.5]. Hence, G is a 2-oscillating group. Then the statement follows directly from Remark 2.1 and Theorem 3.5. □

A paratopological group G is called a *paratopological SIN-group* [9] (*paratopological LSIN-group* [3], respectively) if, for each neighborhood U of identity e of G, there is a neighborhood $W \subseteq G$ of e such that $g^{-1}Wg \subseteq U$ for each $g \in G$ (for each $g \in W$, respectively). It is clear that each topological group and each paratopological SIN-group are paratopological LSIN-groups. Since 2-oscillating paratopological groups contain all saturated paratopological groups and paratopological LSIN-groups [3, Proposition 3], Theorem 3.5 implies the following result.

Corollary 3.8. *Every Hausdorff saturated paratopological group (or paratopological LSIN-group) G with $Hs(G) \cdot \psi(G) \leq \omega$ is submetrizable. In addition, if G is ω-balanced, then G admits a weaker metrizable topological group topology.*

Corollary 3.9. *Every Hausdorff locally countable saturated paratopological group (or paratopological LSIN-group) G is submetrizable. In addition, if G is ω-balanced, then G admits a weaker metrizable topological group topology.*

Proof. According to Corollary 3.8, it is enough to show that $Hs(G) \cdot \psi(G) \leq \omega$. Since G is locally countable, there exists an open neighborhood U at the identity of G such that U is a countable set. Then UU^{-1} is also a countable set, say $UU^{-1} = \{x_n | n \in \omega\}$. Since G is Hausdorff, for each point $x_n \in UU^{-1} \setminus \{e\}$, one can find an open neighborhood V_{x_n} at e such that $V_{x_n} \subseteq U$ and $x_n \notin V_{x_n} V_{x_n}^{-1}$. Thus, it is obvious that $\{e\} = \bigcap_{x_n \in UU^{-1} \setminus \{e\}} V_{x_n} V_{x_n}^{-1}$, which implies that $Hs(G) \cdot \psi(G) \leq \omega$. $\square$

Remark 3.10. Clearly, every paratopological SIN-group is an LSIN-group and every Hausdorff first-countable paratopological group G is ω-balanced and satisfies $Hs(G) \cdot \psi(G) \leq \omega$. Thus, Corollary 3.8 generalizes [7, Theorems 3.8 and 3.13] and Corollary 3.9 generalizes [7, Theorem 3.15].

Corollary 3.11 ([7, Theorem 3.10])**.** *Every Hausdorff Abelian paratopological group G with countable π-character is submetrizable.*

Proof. It is obvious that G is an ω-balanced 2-oscillating paratopological group. Hence, by Theorem 3.5, it suffices to prove that $Hs(G) \cdot \psi(G) \leq \omega$.

Let $\mathcal{B}$ be a local base at identity e of G and $\mathcal{C} = \{V_n | n \in \omega\}$ a local π-base at e. Take any $x \in G$ such that $x \neq e$. Since G is Hausdorff, there exists $U \in \mathcal{B}$ such that $x \notin UU^{-1}$. Thus, there exists $n_0 \in \omega$ such that $V_{n_0} \subseteq U$, which implies that $x \notin UU^{-1} \supseteq V_{n_0} V_{n_0}^{-1}$. Hence, $\{e\} = \bigcap_{n \in \omega} V_n V_n^{-1}$. This implies that $\psi(G) \leq \omega$.

It suffices to prove that $\{e\} = \bigcap_{V_n \in \mathcal{C}} V_n V_n^{-1} V_n V_n^{-1}$ to show that $Hs(G) \leq \omega$. It is equivalent to prove that $\{e\} = \bigcap_{V_n \in \mathcal{C}} V_n^2 (V_n^2)^{-1}$ since G is an Abelian group. Indeed, take any $x \in G$ such that $x \neq e$. Since G is Hausdorff, there exists $U \in \mathcal{B}$ such that $x \notin UU^{-1}$. Take an element $W \in \mathcal{B}$ such that $W^2 \subseteq U$. Hence, there exists $n_0 \in \omega$ such that $V_{n_0} \subseteq W$. It implies that $x \notin UU^{-1} \supseteq W^2 (W^2)^{-1} \supseteq V_{n_0}^2 (V_{n_0}^2)^{-1}$. This finishes the proof. $\square$

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