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REGULAR BASES AT NON-ISOLATED POINTS AND METRIZATION THEOREMS

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Abstract

In this paper, we define the spaces with a regular base at non-isolated points and discuss some metrization theorems. We firstly show that a space X is a metrizable space, if and only if X is a regular space with a σ-locally finite base at non-isolated points, if and only if X is a perfect space with a regular base at non-isolated points, if and only if X is a β-space with a regular base at non-isolated points. In addition, we also discuss the relations between the spaces with a regular base at non-isolated points and some generalized metrizable spaces. Finally, we give an affirmative answer for a question posed by F. C. Lin and S. Lin in [7], which also shows that a space with a regular base at non-isolated points has a point-countable base.

1. Introduction

The bases of topological spaces occupy a core position in the study of the topological theories and metrization problems, which has produced many kinds of metrization theorems, and establishes a foundation for the topological development [12]. For example, the following is a classic metrization theorem.

THEOREM 1.1. *The following are equivalent for a space X:*

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(1) *X is metrizable;*

(2) *X is a* T_1*-space with a regular base;*

(3) *X is a regular space with a σ-locally finite base.*

In recent years, the theory of regular bases in topological spaces played an important role in generalized metrizable spaces [2, 17]. On the other hand, in the study of the theories of topological spaces, we are mainly concerned with the properties of neighborhoods on non-isolated points, and also discuss the relation between their properties and global properties. For example, a study of spaces with a sharp base, a weakly uniform base or an uniform base at non-isolated points [2, 3, 7] shows that some properties of a non-isolated point set of a topological space will help us discuss the global construction of a space. Especially, a space X with a uniform base at non-isolated points if and only if X is the open and boundary-compact image of a metric space [7]. The most typical example is the spaces obtained from a metrizable space by isolating the points of a subset.

Let $\mathcal{B}$ be a base for a space X. For any $x \in X$, the base $\mathcal{B}$ of X is called *regular* at a point x if, for every neighborhood U of x, there exists an open subset V such that $x \in V \subset U$ and $\{B \in \mathcal{B} : B \cap V \neq \emptyset \text{ and } B \not\subset U\}$ is finite.

By Theorem 1.1, every metric space has a base which is regular at non-isolated points. However, there exists a non-metrizable space with a base which is regular at non-isolated points, see the following Example 1.2.

EXAMPLE 1.2. Let X be the closed unit interval $\mathbb{I} = [0, 1]$ and B a Bernstein subset of I. In other words, B is an uncountable set which contains no uncountable closed subset of I. Endow X with the following topology, i.e., Michael line [15]: G is an open subset for X if and only if $G = U \cup Z$, where U is an open subset of $\mathbb{I}$ with Euclidean topology and $Z \subset B$. Let $\mathcal{B}$ be a base of $\mathbb{I}$ with the Euclidean topology, where $\mathcal{B}$ is regular at every point of $\mathbb{I}$. Then $\mathcal{P} = \mathcal{B} \cup \big\{ \{x\} : x \in B \big\}$ is a base for X and also regular at non-isolated points.

Hence this causes our interests in a study of spaces with a base which is regular at non-isolated points, and the related problems of the metrizability. In this paper, we shall prove that spaces with a regular base at non-isolated points are strictly between the discretizations of metrizable spaces and protometrizable spaces, and we also obtain some metrization theorems which help us to better understand the relation between the properties at non-isolated points and global properties in the study the generalized metrizable spaces.

In this paper all spaces are T_1 unless it is explicitly stated which separation axiom is assumed, and all maps are continuous and onto. By $\mathbb{R}$, $\mathbb{N}$, denote the set of real numbers and positive integers, respectively. For a space X, let $I = I(X) = \{x : x \text{ is an isolated point of } X\}$ and $\mathcal{I}(X) = \big\{ \{x\} :$

$x \in I(X)\big\}$. Let $\mathcal{P}$ be a family of subsets for X, and we denote

$$\begin{aligned} \mathrm{st}\,(x,\mathcal{P}) &= \cup\{P \in \mathcal{P} : x \in P\}, \quad x \in X; \\ \mathrm{st}\,(A,\mathcal{P}) &= \cup\{P \in \mathcal{P} : A \cap P \neq \emptyset\}, \quad A \subset X; \\ \mathcal{P}^m &= \{P \in \mathcal{P} : \text{ if } P \subset Q \in \mathcal{P}, \text{ then } Q = P\}. \end{aligned}$$

Readers may refer to [6, 13] for unstated definitions and terminology.

2. Regular bases at non-isolated points

Definition 2.1. Let $\mathcal{B}$ be a base of a space X. $\mathcal{B}$ is a *regular base*, see e.g. [6] (*regular base at non-isolated points,* resp.) for X if for each (non-isolated, resp.) point $x \in X$, $\mathcal{B}$ is regular at x.

It is obvious that regular bases $\Rightarrow$ regular bases at non-isolated points, but regular bases at non-isolated points$\not\Rightarrow$ regular bases by Example 1.2.

Definition 2.2. Let $\{\mathcal{W}_i\}_{i\in\mathbb{N}}$ be a sequence of open covers of a space X and $\mathcal{I}(X) \subset \bigcup_{i\in\mathbb{N}} \mathcal{W}_i$. $\{\mathcal{W}_i\}_{i\in\mathbb{N}}$ is called a *strong development,* see e.g. [6](*strong development at non-isolated points,* resp.) for X if for every $x \in X$ $(x \in X - I)$ and each neighborhood U of x there exist a neighborhood V of x and an $i \in \mathbb{N}$ such that $\mathrm{st}\,(V,\mathcal{W}_i) \subset U$. If $\{\mathcal{W}_i\}_{i\in\mathbb{N}}$ is a strong development at non-isolated points, then so is $\big\{\mathcal{W}_i \cup \mathcal{I}(X)\big\}_{i\in\mathbb{N}}$.

The following Lemma 2.3 is proved similarly to Lemma 5.4.3 in [6], and leave to the reader the easy proofs of Lemma 2.4 and 2.5.

Lemma 2.3. *If $\mathcal{B}$ is a regular base at non-isolated points for a space X, then the family $\mathcal{B}^m \subset \mathcal{B}$ is locally finite at non-isolated points and also covers $X - I$.*

Lemma 2.4. *Let $\mathcal{B}$ be a regular base at non-isolated points for X. If $\mathcal{B}' \subset \mathcal{B}$ is point-finite at non-isolated points, then $\mathcal{B}'' = (\mathcal{B} - \mathcal{B}') \cup \mathcal{I}(X)$ is a regular base at non-isolated points for X.*

Lemma 2.5. *If $\mathcal{B}$ is a regular base at non-isolated points for X, put*

$$\mathcal{B}_1 = \mathcal{B}^m, \quad \mathcal{B}_i = \left[\left(\mathcal{B} - \bigcup_{j=1}^{i-1} \mathcal{B}_j\right) \cup \mathcal{I}(X)\right]^m, i = 2, 3, \ldots .$$

Then $\mathcal{B} = \big(\bigcup_{i=1}^{\infty} \mathcal{B}_i\big) \cup \mathcal{I}(X)$, and for each $i \in \mathbb{N}$, $\mathcal{B}_i$ is locally finite at non-isolated points and $\mathcal{B}_{i+1} \cup \mathcal{I}(X)$ refines $\mathcal{B}_i \cup \mathcal{I}(X)$.

Recall that a topological space X is *monotonically normal* [10] if for each ordered pair (p, C), where C is a closed set for X and $p \in X - C$, there exists an open subset $H(p, C)$ satisfying the following conditions:

(i) $p \in H(p, C) \subset X - C$;

(ii) For every closed subset D for X, if $D \subset C$, then $H(p, C) \subset H(p, D)$;

(iii) If $p \neq q \in X$, then $H\big(p, \{q\}\big) \cap H\big(q, \{p\}\big) = \emptyset$.

A T_2-paracompact space or monotonically normal space is a collectionwise normal space [10].

LEMMA 2.6. *If a space X has a strong development at non-isolated points, then X is a monotonically normal and paracompact space.*

PROOF. Let $\{\mathcal{W}_i\}_{i\in\mathbb{N}}$ be a strong development at non-isolated points for X, where $\mathcal{W}_{i+1}$ refines $\mathcal{W}_i$ for every $i \in \mathbb{N}$.

(1) Claim. Let A be a closed subset for X. If $x \in (X - A) \cap (X - I)$, then there exists an $i \in \mathbb{N}$ such that $\mathrm{st}\,(x, \mathcal{W}_i) \cap \mathrm{st}\,(A, \mathcal{W}_i) = \emptyset$.

In fact, since $X - A$ is an open neighborhood of x, there exists a $j \in \mathbb{N}$ and an open neighborhood V of x such that $\mathrm{st}\,(V, \mathcal{W}_j) \subset X - A$. Also, there exists a $i \geqq j$ such that $\mathrm{st}\,(x, \mathcal{W}_i) \subset V$. Since $\mathrm{st}\,(A, \mathcal{W}_i) \subset X - V$, we have $\mathrm{st}\,(x, \mathcal{W}_i) \cap \mathrm{st}\,(A, \mathcal{W}_i) = \emptyset$.

(2) X is a monotonically normal space.

Let C be a closed subset for X and $p \in X - C$. If $p \in I$, then we let $H(p, C) = \{p\}$; if $p \in X - I$, then there exists a minimum $n \in \mathbb{N}$ such that $\mathrm{st}\,(p, \mathcal{W}_n) \cap \mathrm{st}\,(C, \mathcal{W}_n) = \emptyset$ by (1), so we let $H(p, C) = \mathrm{st}\,(p, \mathcal{W}_n)$. Then $H(p, C)$ is an open subset for X. Clearly this definition of $H(p, C)$ satisfies the conditions (i) and (ii) in the above definition of monotonically normal spaces. We next prove that it also satisfies (iii). In fact, for any distinct points p, q in $X - I$, fix the n, m for which:

$$H\big(p, \{q\}\big) = \mathrm{st}\,(p, \mathcal{W}_n) \quad \text{and} \quad H\big(q, \{p\}\big) = \mathrm{st}\,(q, \mathcal{W}_m).$$

Then

$$\mathrm{st}\,(p, \mathcal{W}_n) \cap \mathrm{st}\,(q, \mathcal{W}_n) = \emptyset \quad \text{and} \quad \mathrm{st}\,(p, \mathcal{W}_m) \cap \mathrm{st}\,(q, \mathcal{W}_m) = \emptyset.$$

By the choice of n, m, we have $n = m$, i.e, $H\big(p, \{q\}\big) \cap H\big(q, \{p\}\big) = \emptyset$. Hence it also satisfies (iii) in the definition of monotonically normal spaces.

(3) X is a paracompact space.

Let $\{G_s\}_{s\in S}$ be an open cover for X and $S_0 = \big\{ s \in S : G_s \cap (X - I) \neq \emptyset \big\}$. Fix a well-order by "$<$" on S_0. For every $i \in \mathbb{N}$, $s \in S_0$, put

$$F_{s,i} = X - \left(\mathrm{st}\,(X - G_s, \mathcal{W}_i) \cup \left(\bigcup_{s' < s} G_{s'}\right)\right),$$

then $F_{s,i} \subset G_s$.

(3.1) The closed family $\{F_{s,i}\}_{s\in S_0,\ i\in\mathbb{N}}$ covers $X - I$.

Indeed, for every $x \in X - I$, there exists a minimum $s(x) \in S_0$ such that $x \in G_{s(x)}$. Since $\{\mathcal{W}_i\}_{i\in\mathbb{N}}$ is a strong development at non-isolated points for X, there exists an $i(x) \in \mathbb{N}$ such that $\operatorname{st}\big(x, \mathcal{W}_{i(x)}\big) \subset G_{s(x)}$. Hence $x \in F_{s(x),i(x)}$.

(3.2) For every $i \in \mathbb{N}$, $\{F_{s,i}\}_{s\in S_0}$ is a discrete and closed family for X.

The family $\{F_{s,i}\}_{s\in S_0}$ is disjoint by construction, hence if $x \in I$ then $\{x\}$ is a neighborhood that intersects $F_{s,i}$ for at most one s. If $x \in X \setminus I$ then, using (3.1), $x \in \bigcup_{s\in S_0} G_s$. Hence there exists a minimum $s(x) \in S_0$ such that $x \in G_{s(x)}$. Then $G_{s(x)} \cap \operatorname{st}(x, \mathcal{W}_i)$ is an open neighborhood of x. If $s' < s(x)$, then $x \in X - G_{s'}$, so we have

$$\operatorname{st}(x, \mathcal{W}_i) \subset \operatorname{st}(X - G_{s'}, \mathcal{W}_i) \quad \text{and} \quad \operatorname{st}(x, \mathcal{W}_i) \cap F_{s',i} = \emptyset.$$

If $s' > s(x)$, then $G_{s(x)} \cap F_{s',i} = \emptyset$, so there is only one member of $\{F_{s,i}\}_{s\in S_0}$ which meets $G_{s(x)} \cap \operatorname{st}(x, \mathcal{W}_i)$. Hence $\{F_{s,i}\}_{s\in S_0}$ is a discrete and closed family for X.

X is collectionwise normal since monotonically normal spaces are collectionwise normal [10]. For every $F_{s,i}$, there exists an open subset $G_{s,i}$ such that $F_{s,i} \subset G_{s,i} \subset G_s$ and $\{G_{s,i}\}_{s\in S_0}$ is a discrete family. Let

$$\mathcal{B}_i = \{G_{s,i}\}_{s\in S_0} \cup \left\{ \{x\} : x \in I - \bigcup_{s\in S_0} G_{s,i} \right\}.$$

Then $\bigcup_{i\in\mathbb{N}} \mathcal{B}_i$ is a σ-locally finite open cover for X and refines $\{G_s\}_{s\in S}$. Since X is regular, X is paracompact. □

Next we shall prove the main theorems in this section.

Theorem 2.7. *A space X has a regular base at non-isolated points if and only if X has a strong development at non-isolated points.*

Proof. Necessity. Since X has a regular base at non-isolated points, X has a regular base at non-isolated points $\mathcal{B} = \big(\bigcup_{i\in\mathbb{N}} \mathcal{B}_i\big) \cup \mathcal{I}(X)$ satisfying Lemma 2.5, where $\mathcal{B}_i$ is locally finite at non-isolated points and $\mathcal{B}_{i+1} \cup \mathcal{I}(X)$ refines $\mathcal{B}_i \cup \mathcal{I}(X)$ for every $i \in \mathbb{N}$. Put $\mathcal{W}_i = \mathcal{B}_i \cup \mathcal{I}(X)$. We will show that $\{\mathcal{W}_i\}_{i\in\mathbb{N}}$ is a strong development at non-isolated points for X. In fact, for every $x \in X - I$ and each open neighborhood U of x, since $\mathcal{B}$ is regular at non-isolated points, there exists an open neighborhood $V \subset U$ of x such that the set of all members of $\mathcal{B}$ that meet both V and $X - U$ is finite. We can denote these finite elements by $B_1, B_2, \ldots, B_k$. Then there exists a $j \in \mathbb{N}$ such that $\mathcal{B}_j \cap \{B_i : i \leqq k\} = \emptyset$. Hence $\operatorname{st}(V, \mathcal{W}_j) \subset U$.

Sufficiency. Let $\{\mathcal{W}_i\}_{i\in\mathbb{N}}$ be a strong development at non-isolated points for X. By Lemma 2.6, X is paracompact. For every $i\in\mathbb{N}$, let $\mathcal{B}_i$ be a locally finite open refinement for $\mathcal{W}_i$. Without loss of generality, we may assume $\mathcal{B}_{i+1}$ refines $\mathcal{B}_i$ for every $i\in\mathbb{N}$. We next prove that $\mathcal{B}=\left(\bigcup_{i\in\mathbb{N}}\mathcal{B}_i\right)\cup\mathcal{I}(X)$ is a regular base at non-isolated points for X. Obviously $\mathcal{B}$ is a base for X. For every $x\in X-I$ and each open neighborhood U of x, there exist an open neighborhood V of x and an $i\in\mathbb{N}$ such that $\operatorname{st}(V,\mathcal{W}_i)\subset U$. If $j\geqq i$, then

$$\operatorname{st}(V,\mathcal{B}_j)\subset\operatorname{st}(V,\mathcal{B}_i)\subset\operatorname{st}(V,\mathcal{W}_i)\subset U.$$

However, since each $\mathcal{B}_j$ is locally finite, there exists an open neighborhood $W(x)$ of x such that the set of all members of $\bigcup_{j<i}\mathcal{B}_j$ that meet $W(x)$ is finite. Let$V_1=V\cap W(x)$. Then the set of all members of $\mathcal{B}$ that meet V_1 and $X-U$ is finite. □

Similar to definition 2.2, we say a space X has a *development at non-isolated points* [7] if there exists a sequence $\{\mathcal{W}_i\}_{i\in\mathbb{N}}$ of open covers for X such that, for every $x\in X-I$ and each open neighborhood U of x, there exist an open neighborhood V of x and an $i\in\mathbb{N}$ such that $\operatorname{st}(V,\mathcal{W}_i)\subset U$.

THEOREM 2.8. *A space X has a regular base at non-isolated points if and only if X is a T_2-paracompact space with a development at non-isolated points.*

PROOF. Necessity. By Lemma 2.6 and Theorem 2.7, if X has a regular base at non-isolated points, then X is a T_2-paracompact space with a development at non-isolated points.

Sufficiency. Let X be a T_2-paracompact space with a development $\{\mathcal{W}_i\}_{i\in\mathbb{N}}$ at non-isolated points. Since X is a T_2-paracompact space, there exists a sequence of open covers $\{\mathcal{B}_i\}_{i\in\mathbb{N}}$ for X such that $\mathcal{B}_{i+1}$ is a star refinement of $\mathcal{B}_i\wedge\mathcal{W}_{i+1}$ for every $i\in\mathbb{N}$. We next prove that $\{\mathcal{B}_i\}_{i\in\mathbb{N}}$ is a strong development at non-isolated points for X. For every $x\in X-I$ and every open neighborhood U of x, there exists an $i\in\mathbb{N}$ such that $\operatorname{st}(x,\mathcal{W}_i)\subset U$. Choose a $V\in\mathcal{B}_{i+1}$ such that $x\in V$. Then

$$\operatorname{st}(V,\mathcal{B}_{i+1})\subset\operatorname{st}(x,\mathcal{B}_i)\subset\operatorname{st}(x,\mathcal{W}_i)\subset U.$$

By Theorem 2.7, X has a regular base at non-isolated points. □

REMARK 2.9. We cannot omit the condition "T_2" in Theorem 2.8. In fact, let X be the finite complement topology on $\mathbb{N}$. Then X is a T_1-compact and developable space, but it is not a T_2-space.

The following corollary is a complement for Lemma 2.5.

COROLLARY 2.10. *A space X has a regular base at non-isolated points if and only if X is a regular space with a development at non-isolated points $\left\{\mathcal{B}_i\cup\mathcal{I}(X)\right\}_{i\in\mathbb{N}}$, where $\mathcal{B}_i$ is locally finite at non-isolated points for every $i\in\mathbb{N}$.*

PROOF. Necessity. It is easy to see by the proof of necessity in Theorems 2.7 and 2.8.

Sufficiency. Let X be a regular space with a development at non-isolated points $\{\mathcal{B}_i \cup \mathcal{I}(X)\}_{i\in\mathbb{N}}$, where $\mathcal{B}_i$ is locally finite at non-isolated points for every $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let

$$U_i = \{x \in X : \mathcal{B}_i \text{ is locally finite at point } x\}.$$

Then U_i is an open subset and $\mathcal{B}_i$ is locally finite at each point of U_i. Since $X - I \subset U_i$, $X - U_i \subset I$ and $X - U_i$ is an open subset for X. Hence U_i is an open and closed subset for X. Thus $\mathcal{B}_i \mid U_i = \{B \cap U_i : B \in \mathcal{B}_i\}$ is an open and locally finite family.

By Theorem 2.8, we only need to prove that X is a paracompact space. In fact, for every open cover $\mathcal{U}$ of X and each $i \in \mathbb{N}$, let

$$\mathcal{V}_i = \{B \cap U_i : B \in \mathcal{B}_i \text{ and there exists an } U \in \mathcal{U} \text{ such that } B \subset U\}$$

and

$$V_i = \cup \mathcal{V}_i.$$

Put

$$\mathcal{V} = \Big(\bigcup_{i\in\mathbb{N}} \mathcal{V}_i\Big) \cup \big\{\{x\} : x \in F\big\}, \quad \text{where} \quad F = \bigcap_{i\in\mathbb{N}} (X - V_i).$$

Then $\mathcal{V}$ is a cover for X and $F \subset I$. In fact, if $x \in X - I$, then there exists an $U \in \mathcal{U}$ such that $x \in U$. Hence there exists an $n \in \mathbb{N}$ such that $\mathrm{st}\,(x, \mathcal{B}_n) \subset U$. Fix a $B \in \mathcal{B}_n$ such that $x \in B$. Then $B \subset U$ and $x \in B \cap U_n \in \mathcal{V}_n$. So $x \in V_n$. Then F is a closed and discrete subset for X. Hence $\mathcal{V}$ is an open σ-locally finite cover and refines $\mathcal{U}$. By the regularity, X is a paracompact space. □

EXAMPLE 2.11. There exists a non-regular T_2-space with a development at non-isolated points.

Let $\mathbb{Q}$, $\mathbb{P}$ denote the rational numbers and the irrational numbers, respectively. Let $X = \mathbb{R}$ and endow X with the following topology [4]: every point of $\mathbb{P}$ is an isolated point; every point $x \in \mathbb{Q}$ has neighborhoods of the following form:

$$B(x, n) = \{x\} \cup \big\{ y \in \mathbb{P} : |y - x| < 1/n \big\}, \quad n \in \mathbb{N}.$$

Then X is a non-regular T_2-space and the isolated points set of X is $\mathbb{P}$. We denote $\mathbb{Q} = \{q_m : m \in \mathbb{N}\}$. For any $n, m \in \mathbb{N}$, let

$$\mathcal{B}_{n,m} = \big\{ B(q_m, n), \mathbb{R} - \{q_m\} \big\},$$

Then $\mathcal{B}_{n,m}$ is a finite open cover for X, and $\mathrm{st}\big(q_m, \mathcal{B}_{n,m} \cup \mathcal{I}(X)\big) = B(q_m, n)$. Hence $\big\{\mathcal{B}_{n,m} \cup \mathcal{I}(X)\big\}_{n,m\in\mathbb{N}}$ is a development at non-isolated points for X and $\mathcal{B}_{n,m}$ is locally finite for any $n, m \in \mathbb{N}$.

3. Metrization theorems

In this section we shall discuss the metrization problems on spaces with the properties of bases at non-isolated points.

X is called a *perfect space* if every open subset of X is an F_σ-set in X.

THEOREM 3.1. *Let X be a space. Then the following are equivalent:*

(1) *X is metrizable;*

(2) *X is a perfect space with a regular base at non-isolated points;*

(3) *X is a perfect space with a strong development at non-isolated points.*

PROOF. By Theorems 1.1 and 2.7, we only need to prove (3) $\Rightarrow$ (1).

Let X be a perfect space with a strong development at the non-isolated points $\{\mathcal{W}_i\}_{i\in\mathbb{N}}$ of X. Then there exists a sequence of open sets $\{G_n\}_{n\in\mathbb{N}}$ such that $X - I = \bigcap_{n=1}^{\infty} G_n$. For every $n \in \mathbb{N}$, let $\mathcal{U}_n = \{G_n\} \cup \big\{\{x\} : x \in I - G_n\big\}$. Then $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a sequence of open covers for X. Put $\mathcal{V}_{2n-1} = \mathcal{W}_n$ and $\mathcal{V}_{2n} = \mathcal{U}_n$, for each $n \in \mathbb{N}$. Then $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ is a strong development for X, and X is metrizable by [6, Theorem 5.4.2]. □

REMARK 3.2. By Example 1.2, we see the condition "X is perfect" in (2) and (3) of Theorem 3.1 cannot be omitted, although clearly it can be replaced with the condition that $I(X)$ is an F_σ-set.

DEFINITION 3.3. Let $\mathcal{B} = \bigcup_{i\in\mathbb{N}} \mathcal{B}_i$ be a base for space X. $\mathcal{B}$ is called *σ-locally finite at non-isolated points,* if for every $i \in \mathbb{N}$, $\mathcal{B}_i$ is locally finite at non-isolated points for X.

Similarly, we can define the notion of spaces with a *σ-discrete base at non-isolated points.*

DEFINITION 3.4. Let $\mathcal{B}$ be a family of subsets of X. For every $x \in X$, $\mathcal{B}$ is called *hereditarily closure-preserving at x* if, for any $H(B) \subset B \in \mathcal{B}$, $x \in \overline{\cup\big\{H(B) : B \in \mathcal{B}\big\}}$, then $x \in \cup\big\{\overline{H(B)} : B \in \mathcal{B}\big\}$. $\mathcal{B}$ is called *a hereditarily closure-preserving collection* for X if, for every $x \in X$, $\mathcal{B}$ is hereditarily closure-preserving at x.

It is easy to verify that a collection is hereditarily closure preserving if and only if it is hereditarily closure preserving at non-isolated points.

LEMMA 3.5. *Let $\mathcal{B}$ be locally finite at non-isolated points for X. Then $\mathcal{B}$ is hereditarily closure-preserving.*

Proof. Let $\mathcal{B} = \{B_\alpha : \alpha \in \Gamma\}$. For every $\alpha \in \Gamma$, choose $H_\alpha \subset B_\alpha$. We can assume $x \in X - I$ and denote $\mathcal{H} = \{H_\alpha\}_{\alpha \in \Gamma}$. If $x \in \overline{\cup \mathcal{H}}$, then there exists an open neighborhood $U(x)$ of x such that the set of all members of $\{H_\alpha\}_{\alpha \in \Gamma}$ that meet $U(x)$ is finite because $\{H_\alpha\}_{\alpha \in \Gamma}$ is locally finite at non-isolated points. we denote these finite elements by $H_{\alpha_1}, H_{\alpha_2}, \dots, H_{\alpha_n}$. Since

$$\overline{\cup \mathcal{H}} = \overline{\cup\big(\mathcal{H} - \{H_{\alpha_i} : i \leqq n\}\big)} \cup \overline{\cup \{H_{\alpha_i} : i \leqq n\}}, \quad \text{and}$$

$$U(x) \cap \Big(\cup \big(\mathcal{H} - \{H_{\alpha_i} : i \leqq n\}\big)\Big) = \emptyset,$$

we have $x \in \overline{\cup \{H_{\alpha_i} : i \leqq n\}}$. Hence $x \in \cup \overline{\mathcal{H}}$. □

Lemma 3.6 [5]. *A regular space X is metrizable if and only if X has a σ-hereditarily closure-preserving base.*

Lemma 3.7. *Let X be a regular space. Then the following conditions are equivalent:*

(1) *X is metrizable;*

(2) *X has a base which is σ-discrete at non-isolated points;*

(3) *X has a base which is σ-locally finite at non-isolated points.*

Proof. It is easy to see by Theorem 1.1, Lemmas 3.5 and 3.6 □

Let X be a topological space and $\tau(X)$ its topology. $g : \mathbb{N} \times X \to \tau(X)$ is called a g-function if, for any $x \in X$ and $n \in \mathbb{N}$, $x \in g(n, x)$. A space X is called a *β-space* [11] if there exists a g-function such that, for every $x \in X$ and sequence $\{x_n\}$ in X, if $x \in g(n, x_n)$ for each $n \in \mathbb{N}$, then $\{x_n\}$ has a cluster point in X. Obviously every developable space is a β-space.

Theorem 3.8. *A space X is metrizable if and only if X is a β-space with a regular base at non-isolated points.*

Proof. We only need to prove the sufficiency. Let X be a β-space with a regular base at non-isolated points. By Theorem 3.1, it suffices to prove that $I(X)$ is an F_σ-set. Suppose g is a g-function satisfying the above definition of β-spaces. Since X has a regular base at non-isolated points, X has a regular base at non-isolated points $\mathcal{B} = \big(\bigcup_{n \in \mathbb{N}} \mathcal{B}_n\big) \cup \mathcal{I}(X)$ satisfying Lemma 2.5, where $\mathcal{B}_n$ is locally finite at non-isolated points and $\mathcal{B}_{n+1} \cup \mathcal{I}(X)$ refines $\mathcal{B}_n \cup \mathcal{I}(X)$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and $x \in X - I$, put

$$b(n, x) = \cap \{B \in \mathcal{B}_n : x \in B\}.$$

Then $\big\{ b(n, x) \big\}_{n \in \mathbb{N}}$ is a local base for $x \in X - I$. For each $n \in \mathbb{N}$, put

$$h(n, x) = \Big(\cap \big\{ g(i, x) : i \leqq n \big\} \Big) \cap b(n, x), \quad x \in X - I;$$

$$H_n = \cup\big\{\, h(n,x) : x \in X - I \big\}.$$

Then $X - I \subset H_n$ and H_n is an open subset for X. We next prove $X - I = \bigcap_{n\in\mathbb{N}} H_n$. Let $x \in \bigcap_{n\in\mathbb{N}} H_n$. Then there exists some point $x_n \in X - I$ such that $x \in h(n, x_n)$ for each $n \in \mathbb{N}$. Since X is a β-space and $x \in g(n, x_n)$, $\{x_n\}$ has a cluster point in X. Let y be a cluster point of $\{x_n\}$. Then $y \in X - I$ and $b(n, y)$ is an open neighborhood of y. Without loss of generality, we can assume $x_{n_i} \in b(i, y)$ for each $i \in \mathbb{N}$. We will show that $b(i, x_{n_i}) \subset b(i, y)$. If not, choose a point $z \in b(i, x_{n_i}) - b(i, y)$, then there exists a $B \in \mathcal{B}_i$ such that $y \in B$ and $z \notin B$. Since $x_{n_i} \in b(i, y) \subset B$, $z \in b(i, x_{n_i}) \subset B$, a contradiction. Hence

$$x \in \bigcap_{i\in\mathbb{N}} h(n_i, x_{n_i}) \subset \bigcap_{i\in\mathbb{N}} h(i, x_{n_i}) \subset \bigcap_{i\in\mathbb{N}} b(i, y) = \{y\},$$

i.e, $x = y \in X - I$. Thus $X - I = \bigcap_{n\in\mathbb{N}} H_n$, and I is an F_σ-set for X. By Theorem 3.1, X is metrizable. □

Remark 3.9. The Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ is a β-space, but it is not a perfect space [6, Corollary 3.6.15]; Sorgenfrey line is a perfect space, but it is not a β-space [11, Example 4.4]. Hence, Theorem 3.1 and Theorem 3.8 are independent each other.

4. Relations with generalized metrizable spaces

Definition 4.1 [14]. Let X be a topological space and let A be a subset of X. The *discretization* of X by A is the space whose topology is generated by the base $\{U : U$ is an open subset of $X\} \cup \big\{\{x\} : x \in A\big\}$. It is denoted by X_A in [6, Example 5.1.22]. We say that a space Y is a *discretization* of X if $Y = X_A$ for some $A \subset X$.

Theorem 4.2. *Let X be a metric space. If $A \subset X$ and X_A is the discretization of X by A, then X_A has a regular base at non-isolated points.*

Proof. Since X is a metric space, X has a regular base $\mathcal{B}_1$. Let $\mathcal{B} = \mathcal{B}_1 \cup \big\{\{x\} : x \in A\big\}$. Obviously, $\mathcal{B}$ is a regular base at non-isolated points for X_A. □

Remark 4.3. If a space X with a regular base at non-isolated points, then is it a discretizable space of a metric space? The answer is negative, see Example 4.4. Recall that X is said to have a G_δ-*diagonal* if there exists a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers such that $\{x\} = \bigcap_{n\in\mathbb{N}} \mathrm{st}\,(x, \mathcal{U}_n)$ for every $x \in X$.

Example 4.4. There exists a space Y having a regular base at non-isolated points. However, Y is not a discretization of a metric space.

Let X be the Michael line in Example 1.2, and denote it by X_B. Let X^* be a copy of X_B and $f : X_B \to X^*$ a homeomorphic map. Put $Z = X_B \bigoplus X^*$ and let $g : Z \to Y$ be a quotient map by identifying $\{x, f(x)\}$ to a point for each $x \in X_B \setminus B$ in Z. Then Y is a quotient space.

By [16], it is easy to see Y has no G_δ-diagonal. Since the discretization of a metric space has a G_δ-diagonal, Y is not a discretization of a metric space. We next prove that Y has a regular base at non-isolated points.

Put $\mathcal{I} = \{\{x\} : x \in B\}$ and let $\mathcal{B}$ be a regular base of $\mathbb{I}$ with the Euclidean topology. Then $\mathcal{B} \cup \mathcal{I}$ is a regular base at non-isolated points for X_B. Hence $f(\mathcal{B}) \cup f(\mathcal{I})$ is a regular base at non-isolated points for X^*. Then $\mathcal{G} = \{g(B \cup f(B)) : B \in \mathcal{B}\} \cup \mathcal{I} \cup f(\mathcal{I})$ is a regular base at non-isolated points for Y.

Indeed, it is easy to see that $\mathcal{G}$ is a base for Y. For every $y \in Y - I(Y)$ and each open neighborhood U of y in Y, there exists a point $x \in X_B$ such that $g(x) = y$. Then $g(f(x)) = y$, and x, $f(x) \in g^{-1}(U)$. Since

$$\mathcal{B}_0 = \mathcal{B} \cup f(\mathcal{B}) \cup \mathcal{I} \cup f(\mathcal{I})$$

is a regular base at non-isolated points for Z, there exist open neighborhoods V_x, $V_{f(x)} \subset g^{-1}(U)$ of x, $f(x)$ in Z respectively such that the set of all members of $\mathcal{B}_0$ that meet V_x and $Z - g^{-1}(U)$ is finite, and the set of all members of $\mathcal{B}_0$ that meet $V_{f(x)}$ and $Z - g^{-1}(U)$ is also finite. Since f is a homeomorphic map, there exists a $B \in \mathcal{B}$ such that $x \in B \subset V_x$ and $f(x) \in f(B) \subset V_{f(x)}$. Then $g(x) = y \in g(B \cup f(B)) \subset U$. Since the set of all members of $\mathcal{B}_0$ that meet $B \cup f(B)$ and $Z - g^{-1}(U)$ is finite. If $V \in \mathcal{B}_0$, then $g^{-1}(g(V)) = V$, hence the set of all members of $\mathcal{G}$ that meet $g(B \cup f(B))$ and $Y - U$ is finite. Thus Y has a regular base at non-isolated points.

Definition 4.5 [14]. An *ortho-base* $\mathcal{B}$ for X is a base of X such that either $\cap\mathcal{A}$ is open in X or $\cap\mathcal{A} = \{x\} \notin \mathcal{I}(X)$ and $\mathcal{A}$ is a neighborhood base at x in X for each $\mathcal{A} \subset \mathcal{B}$. A space X is a *proto-metrizable space* if it is a paracompact space with an ortho-base.

Recall that a space X is called a *γ-space* if there exists a g-function $g(n, x)$ for X satisfying for each $x \in X$ and sequences $\{x_n\}$, $\{y_n\}$ if $x_n \in g(n, y_n)$ and $y_n \in g(n, x)$ for each $n \in \mathbb{N}$, then $x_n \to x$.

Theorem 4.6. *If a space X has a regular base at non-isolated points, then X is:*

(1) *a proto-metrizable space, and*

(2) *a γ-space.*

Proof. (1) By Lemma 2.6 and Theorem 2.7, X is a paracompact space. Also, X has an ortho-base by [7, Theorem 3.4]. Hence X is a proto-metrizable space.

(2) To prove part (2), for each $n \in \mathbb{N}$ and $x \in X$ define a function $g : \mathbb{N} \times X \to \tau(X)$ as follows: if $x \in I$, then $g(n,x) = \{x\}$; if $x \in X - I$, then $g(n,x) = b(n,x)$, where $b(n,x)$ is the same as in the proof in Theorem 3.8. Then $\left\{ g(n,x) \right\}_{n\in\mathbb{N}}$ is a decreasing and open neighborhood base of x, and if $y \in g(n,x)$, then $g(n,y) \subset g(n,x)$. For each $x \in X$ and sequences $\{x_n\}$, $\{y_n\}$, if $x_n \in g(n,y_n)$ and $y_n \in g(n,x)$ for each $n \in \mathbb{N}$, then $x_n \in g(n,y_n) \subset g(n,x)$, thus $x_n \to x$. Hence X is a γ-space. □

Example 4.7. There exists a proto-metrizable space which has no regular base at non-isolated points.

The proto-metrizable but non-γ-space described in Section 3 in [9] works.

Remark 4.8. From the discussion above, it can be seen that spaces with a regular base at non-isolated points are strictly between the discretizations of metrizable spaces and proto-metrizable spaces.

Corollary 4.9. *Let X have a G_δ-diagonal. Then the following conditions are equivalent:*

(1) *X is a discretizations of a metrizable space;*

(2) *X has a regular base at non-isolated points;*

(3) *X is a proto-metrizable space.*

Proof. By Theorems 4.2 and 4.6, we have (1) ⇒ (2) ⇒ (3). By [9, Theorem 3.1], it can be obtained (3) ⇒ (1). □

The condition "G_δ-diagonal" cannot be omitted in Corollary 4.9 by Example 4.4.

Question 4.10. *Under what conditions a proto-metrizable space has a regular base at non-isolated points?*

Remark 4.11. Since a proto-metrizable space is a paracompact space, Theorem 2.8 is an answer for Question 4.10. However, we expect a simpler answer.

Definition 4.12. Let $\mathcal{B}$ be a base of a space X. $\mathcal{B}$ is *point-regular* [1] (*point-regular at non-isolated points* [7], resp.) for X, if for each (non-isolated, resp.) point $x \in X$ and $x \in U$ with U open in X, $\{B \in \mathcal{B} : x \in B \not\subset U\}$ is finite.

Obviously, every regular base at non-isolated points is a point-regular base at non-isolated points. In [7], it is proved that a space X has a point-regular base at non-isolated points if and only if X is an open, boundary-compact image of a metric space. On the other hand, a space X is an open, boundary-compact, s-image of a metric space if and only if X has a point-countable base which is point-regular at non-isolated points. The following question is posed in [7, Question 5.1]:

QUESTION 4.13 (see [7, Question 5.1]). *Let a space X have a point-countable base. If X has a point-regular base at nos-isolated points, is X an open, boundary-compact, s-image of a metric space?*

Next, we give an affirmative answer for Question 4.13.

A space X is called *metalindelöf* if every open cover of X has a point-countable open refinement.

THEOREM 4.14. *The following are equivalent for a space X:*

(1) *X has a point-countable base, and has a point-regular base at non-isolated points;*

(2) *X has a point-countable base which is point-regular at non-isolated points;*

(3) *X is an open boundary-compact, s-image of a metric space;*

(4) *X is an open s-image of a metric space, and is an open boundary-compact image of a metric space;*

(5) *X is a metalindelöf space with a point-regular base at non-isolated points.*

PROOF. It is proved in [7] that if $\mathcal{P}$ is a point-regular base at non-isolated points for a space X, then we can assume that $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ satisfies the following conditions:

(a) $\mathcal{P}_n$ is an open cover and is point-finite at non-isolated points;

(b) $\{\mathcal{P}_n\}$ is a development at non-isolated points for X.

(1) $\Rightarrow$ (2). Suppose that X has a point-countable base $\mathcal{B}$, and suppose that X has a point-regular base at non-isolated points $\mathcal{P}$. We can assume that $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ satisfies the conditions (a) and (b). For each $n \in \mathbb{N}$, put

$$\mathcal{B}' = \big\{ B \in \mathcal{B} : B \not\subset I(X) \big\};$$

$$\mathcal{V}_n(B) = \{P \in \mathcal{P}_n : B \subset P\}, \quad \forall\, B \in \mathcal{B}';$$

$$\hat{P} = \cup\big\{ B \in \mathcal{B}' : P \in \mathcal{V}_n(B) \big\}, \quad \forall\, P \in \mathcal{P}_n;$$

$$\hat{\mathcal{P}}_n = \{\hat{P} : P \in \mathcal{P}_n\}.$$

Then $\hat{\mathcal{P}}_n$ is point-countable. In fact, if $x \in \hat{P} \in \hat{\mathcal{P}}_n$, then there is $B' \in \mathcal{B}'$ such that $x \in B'$ and $P \in \mathcal{V}_n(B')$. Since $\{B \in \mathcal{B}' : x \in B\}$ is countable, and each $\mathcal{V}_n(B)$ is finite for each $B \in \mathcal{B}'$ by the condition (a), it follows that $\{ P \in \mathcal{V}_n(B) : x \in B \in \mathcal{B}' \}$ is countable.

Put

$$\hat{\mathcal{P}} = \left(\bigcup_{n \in \mathbb{N}} \hat{\mathcal{P}}_n \right) \cup \mathcal{I}(X).$$

Then $\hat{\mathcal{P}}$ is point-countable. If $x \in U - I$ with U open in X, then there is $m \in \mathbb{N}$ such that $x \in \mathrm{st}\,(x, \mathcal{P}_m) \subset U$ by the condition (b). Take $P \in \mathcal{P}_m$ with $x \in P$, then there is $B \in \mathcal{B}'$ such that $x \in B \subset P$, thus $P \in \mathcal{V}_m(B)$, and $x \in B \subset \hat{P} \subset P \subset U$. So $\hat{\mathcal{P}}$ is a base for X. Finally, it is easy to see that $\hat{\mathcal{P}}$ is point-regular at non-isolated points by $\hat{P} \subset P$ for each $P \in \mathcal{P}$.

(2) $\Rightarrow$ (3) by [7, Corollary, 3.2]. (3) $\Rightarrow$ (4) is obvious. And (4) $\Rightarrow$ (5) by [7, Theorem, 3.1].

(5) $\Rightarrow$ (1). Let X be a metalindelöf space with a point-regular base at non-isolated points. As in the proof of (1) $\Rightarrow$ (2), there is a sequence $\{\mathcal{P}_n\}$ of open covers of X such that $\{\mathcal{P}_n\}$ is a development at non-isolated points for X. For each $n \in \mathbb{N}$, let $\mathcal{B}_n$ be a point-countable open refinement of $\mathcal{P}_n$. And put

$$\mathcal{B} = \left(\bigcup_{n \in \mathbb{N}} \mathcal{B}_n \right) \cup \mathcal{I}(X).$$

Then $\mathcal{B}$ is a point-countable base for X. In fact, if a non-isolated point $x \in U$ with U open in X, then there is $n \in \mathbb{N}$ such that $\mathrm{st}\,(x, \mathcal{P}_n) \subset U$. Take $B \in \mathcal{B}_n$ with $x \in B$, then $x \in B \subset \mathrm{st}\,(x, \mathcal{B}_n) \subset \mathrm{st}\,(x, \mathcal{P}_n) \subset U$. □

By Theorem 4.14, the following is obtained.

Corollary 4.15. *Every space with a regular base at non-isolated points has a point-countable base.*

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