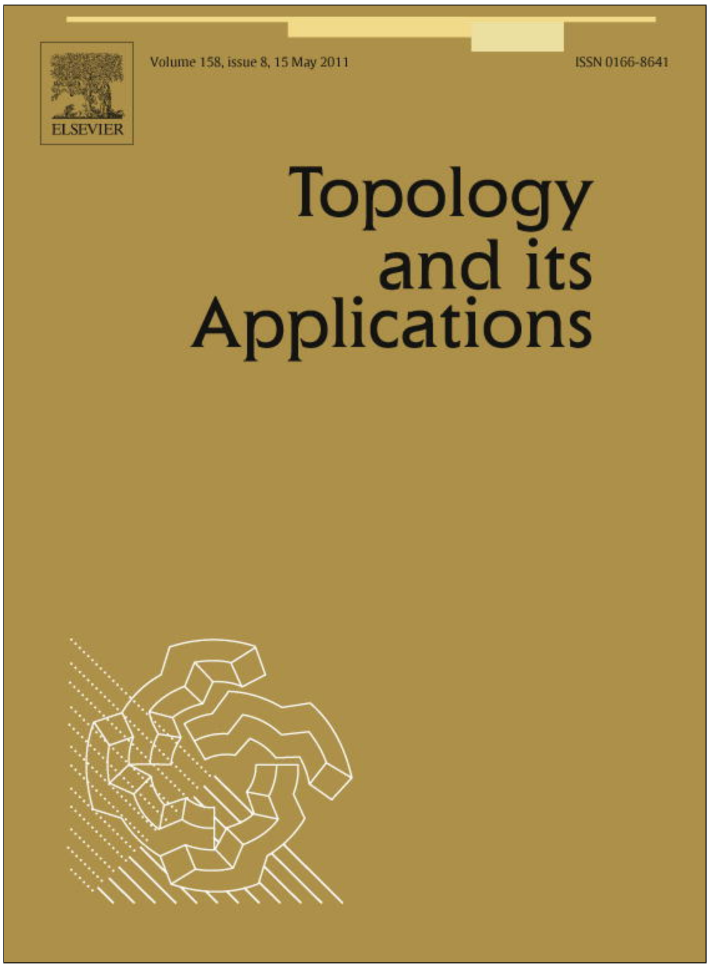
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# Some weak versions of the $M_1$-spaces ☆

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ABSTRACT

We mainly introduce some weak versions of the $M_1$-spaces, and study some properties about these spaces. The mainly results are that: (1) If $X$ is a compact scattered space and $i(X) \leqslant 3$, then $X$ is an $s$-$m_1$-space; (2) If $X$ is a strongly monotonically normal space, then $X$ is an $s$-$m_2$-space; (3) If $X$ is a $\sigma$-$m_3$-space, then $t(X) \leqslant c(X)$, which extends a result of P.M. Gartside (1997) in [7]. Moreover, some questions are posed in the paper.



## 1. Introduction

All spaces are $T_1$ and regular unless stated otherwise, and all maps are continuous and onto. The letter $\mathbb{N}$ denotes the set of all positively natural numbers. Let $X$ be a topological space. Recalled that a family $\mathcal{P}$ of subsets of $X$ is called *closure-preserving* if, for any $\mathcal{P}' \subset \mathcal{P}$, we have $\overline{\bigcup \mathcal{P}'} = \bigcup\{\overline{P}\colon\ P \in \mathcal{P}'\}$. Moreover, $X$ is called an $M_1$-space [3] if $X$ has a $\sigma$-closure-preserving base. It is still a famous open problem (usually called the $M_1 = M_3$ question, see [6]) whether each stratifiable space is $M_1$. M. Ito proved that every $M_3$-space with a closure-preserving local base at each point is $M_1$ [10], and T. Mizokami has just showed that every $M_1$-space has a closure-preserving local base at each point [12]. Therefore, to give a positive answer to $M_1 = M_3$, it is sufficient to prove that each stratifiable space has a closure-preserving local base at each point.

R.E. Buck first introduced and studied the $m_i$ (see Definition 2.4) properties in [1], where he gave some interesting and surprising results about $m_i$-spaces. Recently, A. Dow, R. Martínez and V.V. Tkachuk have also make some study on spaces with a closure-preserving local base at each point (in fact, they call such spaces for *Japanese spaces* in their paper) [4]. In this paper we introduce some weak versions of $M_1$-spaces, and study some properties and relations on these spaces.

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## 2. Preliminaries

**Definition 2.1.** Let $X$ be a topological space and $\mathcal{B}$ a family of subsets of $X$. $\mathcal{B}$ is called a *quasi-base* [6] for $X$ if, for each $x$ and open subset $U$ with $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in \operatorname{int}(B) \subset B \subset U$.

**Definition 2.2.** Let $X$ be a topological space and $\mathcal{P}$ a pair-family for $X$, where, for any $P \in \mathcal{P}$, we denote $P = (P', P'')$. $\mathcal{P}$ is called a *pairbase* [6] if $\mathcal{P}$ satisfies the following conditions:

(1) For any $(P', P'') \in \mathcal{P}$, $P' \subset P''$ and $P'$ is open subset of $X$;
(2) For any $x \in U \in \tau(X)$, there exists $(P', P'') \in \mathcal{P}$ such that $x \in P' \subset P'' \subset U$.

Moreover, a pairbase $\mathcal{P}$ is called a *cushioned* if, for each $\mathcal{P}' \subset \mathcal{P}$, we have $\overline{\bigcup\{P'\colon (P', P'') \in \mathcal{P}'\}} \subset \bigcup\{P''\colon (P', P'') \in \mathcal{P}'\}$.

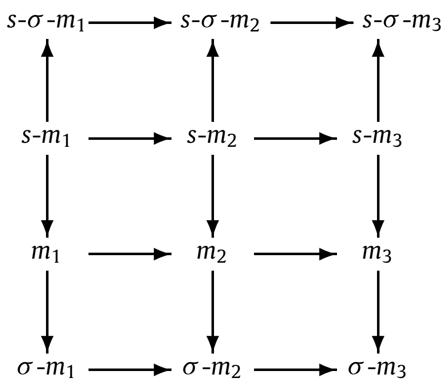
**Definition 2.3.** Let $\mathcal{P}$ be a collection of subsets of $X$. $\mathcal{P}$ is called *closure-preserving* [6] if, for any $\mathcal{P}' \subset \mathcal{P}$, we have $\overline{\bigcup \mathcal{P}'} = \bigcup\{\overline{P}\colon P \in \mathcal{P}'\}$.

A family $\mathcal{A}$ of open subsets of a space $X$ is called *a base of X at a set A* if $A = \bigcap \mathcal{A}$ and for any neighborhood $U$ of $A$, there is a $V \in \mathcal{A}$ such that $A \subset V \subset U$. A family $\mathcal{A}$ of subsets of a space $X$ is called a *quasi-base of X at a set A* if $A = \bigcap \mathcal{A}$ and for any neighborhood $U$ of $A$, there is a $V \in \mathcal{A}$ such that $A \subset \operatorname{int}(V) \subset V \subset U$.

**Definition 2.4.** Let $X$ be a space, $x \in X$ and $F$ a closed subset of $X$. Then

(1) $X$ is $m_1$ ($\sigma$-$m_1$) *at the point x* [1] if $X$ has a closure-preserving ($\sigma$*-closure-preserving*) local base at the point $x$. $X$ is called an $m_1$*-space* ($\sigma$-$m_1$*-space*) if every point of $X$ is $m_1$ ($\sigma$-$m_1$);
(2) $X$ is $s$-$m_1$ ($s$-$\sigma$-$m_1$) *at F* if $X$ has a closure-preserving ($\sigma$*-closure-preserving*) local base at $F$. $X$ is called an $s$-$m_1$*-space* ($s$-$\sigma$-$m_1$*-space*) if every closed subset of $X$ is $m_1$ ($\sigma$-$m_1$);
(3) $X$ is $m_2$ ($\sigma$-$m_2$) *at the point x* [1] if $X$ has a closure-preserving ($\sigma$*-closure-preserving*) local quasi-base at the point $x$. $X$ is called an $m_2$*-space* ($\sigma$-$m_2$*-space*) if every point of $X$ is $m_2$ ($\sigma$-$m_2$);
(4) $X$ is $s$-$m_2$ ($s$-$\sigma$-$m_2$) *at F* if $X$ has a closure-preserving ($\sigma$*-closure-preserving*) local quasi-base at $F$. $X$ is called an $s$-$m_2$*-space* ($s$-$\sigma$-$m_2$*-space*) if every closed subset of $X$ is $m_2$ ($\sigma$-$m_2$);
(5) $X$ is $m_3$ ($\sigma$-$m_3$) *at the point x* [1] if $X$ has a cushioned local pairbase ($\sigma$*-cushioned local pairbase*) at the point $x$. $X$ is called an $m_3$*-space* ($\sigma$-$m_3$*-space*) if every point of $X$ is $m_3$ ($\sigma$-$m_3$);
(6) $X$ is $s$-$m_3$ ($s$-$\sigma$-$m_3$) *at F* if $X$ has a cushioned local pairbase ($\sigma$*-cushioned local pairbase*) at $F$. $X$ is called an $s$-$m_3$*-space* ($s$-$\sigma$-$m_3$*-space*) if every closed subset of $X$ is $m_3$ ($\sigma$-$m_3$).

It is easy to see that




A space $X$ is called *a stratifiable or $M_3$-space* if it has a $\sigma$-cushioned pairbase. By [12], we have that $X$ is $M_1$-space iff $X$ is $M_3$ and $m_1$ iff $X$ is $M_3$ and $s$-$m_1$. Moreover, if we let $X$ be a regular stratifiable space, then all the spaces on the above are equivalent, see [1,10,12].

For each space $X$, we let $I(X)$ be the set of all isolated points of $X$. If $X$ is scattered, then let $X_0 = X$; proceeding inductively assume that $\alpha$ is an ordinal and we constructed $X_\beta$ for all $\beta < \alpha$. If $\alpha = \beta + 1$ for some $\beta$, then we let $X_\alpha = X_\beta \setminus I(X_\beta)$. If $\alpha$ is a limit ordinal, then we let $X_\alpha = \bigcap\{X_\beta\colon \beta < \alpha\}$. The first ordinal $\alpha$ such that $X_\alpha = \emptyset$ is called the *dispersion index of X* and is denoted by $i(X)$, see [4].

Reader may refer to [5,6] for notations and terminology not explicitly given here.

## 3. $s$-$m_1$- and $s$-$\sigma$-$m_1$-spaces

In [4], A. Dow, R. Martínez and V.V. Tkachuk proved that each space with a finite number of non-isolated points is an $m_1$-space, and each compact scattered space $X$ and $i(X) \leqslant 3$ is an $m_1$-space. However, we shall see that there exists an $m_1$-space $X$ such that $X$ is non-$s$-$m_1$, see Example 3.1. But we have the follow Theorems 3.1 and 3.2, which extend the results of the above.

**Lemma 3.1.** *Let $\mathcal{P}$ be a topological property such that* (a) *if space $X$ has $\mathcal{P}$ then $X$ has $m_i$ and* (b) *if $X$ has $\mathcal{P}$ and $A$ is a closed subspace then $X/A$ has $\mathcal{P}$. Then every space $X$ with property $\mathcal{P}$ has the $s$-variant property $s$-$m_i$.*

**Proof.** Take any space $X$ with property $\mathcal{P}$, and take any closed subset $A$ of $X$. Then, by (b), $X/A$ has property $\mathcal{P}$, and so is $m_i$ (by (a)). In particular, the point $A$ in $X/A$ is an $m_i$ point, and so the set $A$ has a 'nice' outer base in $X$. From which it follows that $X$ has $s$-$m_i$. □

**Theorem 3.1.** *Each space with a finite number of non-isolated points is an $s$-$m_1$-space.*

**Proof.** Let $X$ be a space with a finite number of non-isolated points, and let $A$ be a closed subspace of $X$. It is easy to see that $X/A$ has finite number of non-isolated points. By [4, Proposition 2.9], a space with a finite number of non-isolated points is $m_i$, and thus $X$ is an $s$-$m_1$-space by Lemma 3.1. □

**Theorem 3.2.** *If $X$ is a compact scattered space and $i(X) \leqslant 3$, then $X$ is an $s$-$m_1$-space.*

**Proof.** Let $X$ be a compact scattered space and $i(X) \leqslant 3$, and let $A$ be a closed subspace of $X$. It is easy to see that $X/A$ is also a compact scattered space and $i(X) \leqslant 3$. By [4, Theorem 3.1], a compact scattered space and $i(X) \leqslant 3$ is $m_i$, and thus $X$ is an $s$-$m_1$-space by Lemma 3.1. □

The proofs of the following Propositions 3.1, 3.2 and 3.3 are easy, and so we omit them.

**Proposition 3.1.** *If $X$ has a clopen closure-preserving neighborhood base at any closed set then $X$ is hereditarily $s$-$m_1$. In particular, any extremally disconnected $s$-$m_1$-space is hereditarily $s$-$m_1$.*

**Proposition 3.2.** *Suppose that $X$ is a space and a closed set $F \subset X$ has an open neighborhood base in $X$ which is well-ordered by the reverse inclusion. Then $X$ is $s$-$m_1$ at $F$.*

**Proposition 3.3.** *If $X$ is an $s$-$m_1$-space, and $D$ is dense in $X$. Then $D$ is also an $s$-$m_1$-subspace.*

A map $f : X \to Y$ is called *quasi-open* if, for each non-empty open subset $U$ of $X$, the interior of $f(U)$ is non-empty. $f$ is called an *irreducible map* if, for each proper closed subset $F$ of $X$, we have $f(F) \neq Y$.

**Lemma 3.2.** *([11]) Let $f : X \to Y$ be a quasi-open closed map. If $\mathscr{B}$ is a closure-preserving open family of $X$, then $\varphi = \{\mathrm{int}(f(B)): B \in \mathscr{B}\}$ is a closure-preserving open family of $Y$.*

**Theorem 3.3.** *Let $f : X \to Y$ be a quasi-open closed map. If $X$ is an $s$-$m_1$-space, then $Y$ is also an $s$-$m_1$-space.*

**Proof.** Let $F$ be any closed set of $Y$ and $F \neq Y$. Then $f^{-1}(F)$ is closed in $X$. Since $X$ is $s$-$m_1$, $f^{-1}(F)$ has a closure-preserving open neighborhood base $\mathscr{B}$ at $f^{-1}(F)$. Since $f$ is a quasi-open closed map, the family $\varphi = \{\mathrm{int}(f(B)): B \in \mathscr{B}\}$ is a closure-preserving open family of $Y$ by Lemma 3.2. Moreover, because $f$ is a closed map, we have $F \subset \mathrm{int}(f(B))$ for each $B \in \mathscr{B}$. It is easy to see that $\varphi$ is an open neighborhood base at $F$ in $Y$. □

**Corollary 3.1.** *Closed and irreducible maps preserve $s$-$m_1$-spaces.*

**Proof.** Since closed and irreducible maps are quasi-open maps [11], closed and irreducible maps preserve $s$-$m_1$ property by Theorem 3.3. □

Next, we shall give an example to show that there exists an $m_1$-space $X$ which is non-$s$-$m_1$. Firstly, we prove the following Theorem 3.4.

Let $A$ be a subset of a space $X$. We call a family $\mathcal{N}$ of open subsets of $X$ is an *outer base* of $A$ in $X$ if for any $x \in A$ and open subset $U$ with $x \in U$ there is a $V \in \mathcal{N}$ such that $x \in V \subset U$.

**Theorem 3.4.** *If $X$ is Eberlein compact then $X$ is an $s$-$\sigma$-$m_1$-space.*

**Proof.** Let $F$ be any closed subset of $X$. Since $X$ is Eberlein compact, it follows from [4, Theorem 3.13] that $F$ has a $\sigma$-closure-preserving outer base $\mathscr{B} = \bigcup_{n\in\mathbb{N}} \mathscr{B}_n$ in $X$, where, for each $n \in \mathbb{N}$, $\mathscr{B}_n$ is closure-preserving and $\mathscr{B}_n \subset \mathscr{B}_{n+1}$. For each $n \in \mathbb{N}$, let

$$\mathscr{P}_n = \left\{\bigcup \mathscr{B}' \colon\ \mathscr{B}' \text{ is a finite subfamily of } \mathscr{B}_n \text{ and } F \subset \bigcup \mathscr{B}'\right\}.$$

It is easy to see that $\mathscr{P} = \bigcup_{n\in\mathbb{N}} \mathscr{P}_n$ is a $\sigma$-closure-preserving local base at the set $F$ in $X$, where, for each $n \in \mathbb{N}$, $\mathscr{P}_n$ is closure-preserving. □

Recalled that a closed map $f : X \to Y$ which is *perfect* if, for each $y \in Y$, $f^{-1}(y)$ is compact.

**Example 3.1.** There exists an $m_1$-space $X$ such that the following conditions are satisfied:

(1) $X$ is an $s$-$\sigma$-$m_1$-space, and non-$s$-$m_1$-space;
(2) The image of $X$ under some perfect and irreducible map is not an $m_1$-space.

**Proof.** Let $X$ be the Alexandroff double $D$ of the Cantor set $C$. Then $X$ is first countable Eberlein compact space [4]. Hence $X$ is $s$-$\sigma$-$m_1$ by Theorem 3.4. Let $f : X \to Y$ be the quotient map by identifying the non-isolated point of $X$ to one point. Then $f$ is an irreducible and perfect map. However, $Y$ is not an $m_1$-space by [4, Corollary 3.18], and hence $X$ is a not an $s$-$m_1$-space by Corollary 3.1. Moreover, it is easy to see that first-countable spaces are $m_1$. However, $X$ is non-$s$-$m_1$-space. Therefore, compact first-countable is not need to be an $s$-$m_1$-space. □

In [4], the authors prove that each *GO* space is $m_1$. However, we don't know whether each *GO* space is $s$-$m_1$, and so we have the following question.

**Question 3.1.** Let $X$ be a *GO* space. Is $X$ $s$-$m_1$?

## 4. $s$-$m_2$- and $s$-$\sigma$-$m_2$-spaces

Since closed maps preserve a closure-preserving family, we have the following theorem.

**Theorem 4.1.** *Closed maps preserve $s$-$m_2$- and $s$-$\sigma$-$m_2$-spaces, respectively.*

**Theorem 4.2.** *If $X$ is an $s$-$m_2$-space and $Y \subset X$ then $Y$ is $s$-$m_2$.*

A space $X$ is *monotonically normal* if there is a function $G$ which assigns to each closed ordered pair $(H, K)$ of disjoint closed subsets of $X$ an open subset $G(H, K) \subset X$ such that:

(1) $H \subset G(H, K) \subset \overline{G(H, K)} \subset X \setminus K$;
(2) $G(H, K) \subset G(H', K')$ for disjoint closed subsets $H'$ and $K'$ with $H \subset H'$ and $K \supset K'$;
(3) $G(H, K) \cap G(K, H) = \emptyset$.

Moreover, if $X$ also satisfies the following condition:

(4) if $H' \subset G(H, K)$ with $H'$ closed in $X$, then $G(H', K) \subset G(H, K)$,

then $X$ is called *strongly monotonically normal* [8,9].

In [4], the authors pose the following question.

**Question 4.1.** ([4]) Must every monotonically normal space be an $m_1$-space?

Next, we shall give some partial answer for this Question 4.1, see Theorem 4.3.

**Theorem 4.3.** *Let $X$ be a strongly monotonically normal space. Then $X$ is an $s$-$m_2$-space.*

**Proof.** Let $A$ be a closed subspace of $X$. In [2], R.E. Buck, R.W. Heath and P.L. Zenora showed that a closed image of a strongly monotonically normal space is again strongly monotonically normal, and hence $X/A$ is strongly monotonically normal. By [1, Theorem 3.13], a strongly monotonically normal space is $m_2$, and thus $X$ is an $s$-$m_2$-space by Lemma 3.1. □

**Question 4.2.** Let $X$ be a strongly monotonically normal space. Is $X$ an $s$-$m_1$-space or an $m_1$-space?

## 5. $s$-$m_3$- and $\sigma$-$m_3$-spaces

The proofs of the following two theorems are obvious, and so we omit them.

**Theorem 5.1.** *Closed maps preserve $s$-$m_3$- and $s$-$\sigma$-$m_3$-spaces, respectively.*

**Theorem 5.2.** *If $X$ is an $s$-$m_3$-space and $Y \subset X$ then $Y$ is $s$-$m_3$.*

The following theorem is also a partial answer for Question 4.1.

**Theorem 5.3.** *Let $X$ be a monotonically normal space. Then $X$ is an $s$-$m_3$-space.*

**Proof.** Let $A$ be a closed subspace of $X$. It is well known that monotonically normal spaces are preserved by closed images, and implies $m_3$. Hence $X$ is $m_3$ and $X/A$ is monotonically normal, which follows that $X$ is an $s$-$m_3$-space by Lemma 3.1. $\square$

Let $X$ be a space and $\kappa$ an infinite cardinal. For each $x \in X$, we denote $t(x, X)$ means that for any $A \subset X$ with $x \in \overline{A}$ there exists a set $B \subset A$ such that $|B| \leqslant \kappa$ and $x \in \overline{B}$; moreover, $t(X) \leqslant \kappa$ iff $t(x, X) \leqslant \kappa$ for each $x \in X$. The space $X$ with $t(X) \leqslant \kappa$ are said to have tightness $\leqslant \kappa$.

A pairwise disjoint collection of non-empty open subsets in $X$ is called a *cellular family*. The *cellularity of $X$*, defined as follows

$$c(X) = \sup\bigl\{|\mathcal{U}|\colon \mathcal{U} \text{ is a cellular family in } X\bigr\} + \omega.$$

In [7], P.M. Gartside proved that for each monotonically normal space $X$, we have $t(X) \leqslant c(X)$. We shall extend this result of P.M. Gartside, and prove that, for each $\sigma$-$m_3$-space $X$, we have $t(X) \leqslant c(X)$.

**Theorem 5.4.** *Suppose that a space $X$ is $\sigma$-$m_3$ at some point $x \in X$ and $\kappa$ is an infinite cardinal such that $c(U) \leqslant \kappa$ for some open neighborhood $U$ of the point $x$. Then $t(x, X) \leqslant \kappa$. In particular, if $X$ is a $\sigma$-$m_3$-space, then $t(X) \leqslant c(X)$.*

**Proof.** Fix any set $A \subset X \setminus \{x\}$ with $x \in \overline{A}$. Let $\mathscr{P} = \bigcup_{n\in\mathbb{N}} \mathscr{P}_n$ be a $\sigma$-cushioned pairbase at the point $x$, where $\mathscr{P}_n \subset \mathscr{P}_{n+1}$ for each $n \in \mathbb{N}$, and for each $P \in \mathscr{P}$, $P_2$ is closed in $X$. Without loss of generality, we may assume that $A \subset U$ and $\bigcup_{n\in\mathbb{N}}(\bigcup \mathscr{P}_n) \subset U$. For each $y \in X \setminus \{x\}$ and $n \in \mathbb{N}$, put

$$W_{ny} = X \setminus \overline{\bigcup\{P_1\colon\ P \in \mathscr{P} \text{ and } y \notin P_2\}}.$$

For each $n \in \mathbb{N}$, since $y \in X \setminus \bigcup\{P_2\colon\ P \in \mathscr{P} \text{ and } y \notin P_2\} \subset W_{ny}$, $W_{ny}$ is an open neighborhood of $y$.

**Claim 1.** *If $Q \subset X \setminus \{x\}$, then $x \in \overline{Q}$ if and only if, for each $n \in \mathbb{N}$, $x \in \overline{\bigcup\{W_{ny}\colon\ y \in Q\}}$.*

In fact, if $x \in \overline{Q}$, then, for each open neighborhood $V$ of point $x$, we have $V \cap Q \neq \emptyset$. Choose a point $y \in V \cap Q$. It follows that $V \cap W_{ny} \neq \emptyset$ for each $n \in \mathbb{N}$, and hence we have $x \in \overline{\bigcup\{W_{ny}\colon\ y \in Q\}}$. For each $n \in \mathbb{N}$, let $x \in \overline{\bigcup\{W_{ny}\colon\ y \in Q\}}$. Suppose that $x \notin \overline{Q}$. Then there exist an $n \in \mathbb{N}$ and $P \in \mathscr{P}_n$ such that $x \in P_1$ and $P_2 \cap Q = \emptyset$. For each $y \in Q$, since $y \notin P_2$, we have $W_{ny} \cap P_1 = \emptyset$, and so $x \notin \overline{\bigcup\{W_{ny}\colon\ y \in Q\}}$, which is a contradiction.

For each $n \in \mathbb{N}$ and $y \in A$, put $G_{ny} = W_{ny} \cap U$. It follows from $c(U) \leqslant \kappa$ that we can choose a set $D_n \subset A$ such that $|D_n| \leqslant \kappa$ and $G_n = \bigcup\{G_{ny}\colon\ y \in D_n\}$ is dense in $H_n = \bigcup\{G_{ny}\colon\ y \in A\}$. Let $D = \bigcup_{n\in\mathbb{N}} D_n$. Then $|D| \leqslant \kappa$. Obviously, we have that

$$x \in \overline{A} \subset \bigcap_{n\in\mathbb{N}} \overline{H_n} = \bigcap_{n\in\mathbb{N}} \overline{G_n},$$

which implies that $x \in \overline{G_n} \subset \overline{\bigcup\{W_{ny}\colon\ y \in D\}}$ for each $n \in \mathbb{N}$. By Claim 1, we have $x \in \overline{D}$. Therefore, $t(x, X) \leqslant \kappa$. $\square$

**Corollary 5.1.** *([4]) If $X$ is an $m_2$-space, then $t(X) \leqslant c(X)$.*

**Corollary 5.2.** *([7, Theorem 10]) If $X$ is a monotonically normal space, then $t(X) \leqslant c(X)$.*

Recall that a family $\mathcal{U}$ of non-empty open sets of a space $X$ is called a *$\pi$-base* if for each non-empty open set $V$ of $X$, there exists an $U \in \mathcal{U}$ such that $V \subset U$. The *$\pi$-character* of $x$ in $X$ is defined by $\pi_\chi(x, X) = \min\{|\mathcal{U}|: \mathcal{U}$ is a local $\pi$-base at $x$ in $X\}$. The *$\pi$-character of $X$* is defined by $\pi_\chi(X) = \sup\{\pi_\chi(x, X): x \in X\}$.

In [4], A. Dow, R. Ramírez and V.V. Tkachuk proved that if $X$ is a separable $m_2$-space with the Baire property then $X$ has countable $\pi$-character. However, we find the proof has a gap. Next, we shall give out the correct proof. In fact, we have more general result, see Theorem 5.5.

**Theorem 5.5.** *Suppose that a space $X$ is a separable space with the Baire property. If $X$ is $\sigma$-$m_3$ at some point $x \in X$, then it has countable $\pi$-character at the point $x$.*

**Proof.** Suppose that $x$ is a non-isolated point in $X$, and that $D = \{d_n: n \in \mathbb{N}\} \subset X \setminus \{x\}$ is a dense subset of $X$. Let $\mathscr{P} = \bigcup \mathscr{P}_n$ be a $\sigma$-cushioned pairbase at the point $x$, where, for each $n \in \mathbb{N}$, $\mathscr{P}_n$ is cushioned. For each $n \in \mathbb{N}$, put $D_n = D \cap \bigcup\{P_1: P \in \mathscr{P}_n\} = \{d_{n,m}: m \in \mathbb{N}\}$. For each $y \in X \setminus \{x\}$ and $n \in \mathbb{N}$, put

$$W_{ny} = X \setminus \overline{\bigcup\{P_1: P \in \mathscr{P}_n \text{ and } y \notin P_2\}}.$$

Then $W_{ny}$ is an open neighborhood of $y$. For each $n, m \in \mathbb{N}$, put $B_{n,m} = \{y \in X \setminus \{x\}: d_{n,m} \in W_{ny}\}$. It is easy to see that $B_{n,m} \subset \bigcap\{P_2: d_{n,m} \in P_1, P \in \mathscr{P}_n\}$. Further, for each $n, m \in \mathbb{N}$, $B_{n,m}$ is closed in $X \setminus \{x\}$. In fact, let $y \in \overline{B_{n,m}} \setminus \{x\}$. Then $y \in W_{ny}$, and hence $W_{ny} \cap B_{n,m} \neq \emptyset$. For each $z \in W_{ny} \cap B_{n,m}$, we have $W_{nz} \subset W_{ny}$, and thus $d_{n,m} \in W_{nz} \subset W_{ny}$. So $y \in B_{n,m}$. Let $\mathscr{U} = \{\text{int}(B_{n,m}): n, m \in \mathbb{N}\}$. Then $\mathscr{U}$ is a countable $\pi$-base at the point $x$.

In fact, let any open subset $U$ with $x \in U$. Since $\mathscr{P}$ is a local pairbase at the point $x$, there exist an $n \in \mathbb{N}$ and $P \in \mathscr{P}_n$ such that $x \in P_1 \subset P_2 \subset U$, where $P = (P_1, P_2)$. It is easy to see that, for each $m \in \mathbb{N}$, $d_{n,m} \in B_{n,m} \subset P_2$ whenever $d_{n,m} \in P_1$. Let $V = P_1 \setminus \{x\}$. Then $V$ is a non-empty open subset and has the Baire property. If $y \in V$, then $D_n \cap W_{ny} \cap V \neq \emptyset$, and hence there exists an $m \in \mathbb{N}$ such that $d_{n,m} \in W_{ny} \cap V$, i.e., $y \in B_{n,m}$. Therefore, $V \subset \bigcup\{B_{n,m}: d_{n,m} \in P_1\}$. Since $V$ has the Baire property, there exists an $m \in \mathbb{N}$ such that $G = \text{int}(B_{n,m}) \neq \emptyset$. Hence $G \in \mathscr{U}$ and $G \subset B_{n,m} \subset P_2 \subset U$. Therefore, $\mathscr{U}$ is a countable $\pi$-base at the point $x$. □

**Corollary 5.3.** *([4]) Suppose that a space $X$ is a separable space with the Baire property and $X$ is $m_2$ at some point $x \in X$. Then $X$ has countable $\pi$-character at the point $x$.*

**Corollary 5.4.** *If $G$ is a separable $\sigma$-$m_3$-group with the Baire property then $G$ is metrizable.*

**Proof.** Since $w(G) = \pi w(G)$ for any topological group, $G$ is metrizable. □

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## References

[1] R.E. Buck, Some weaker monotone separation and basis properties, Topology Appl. 69 (1996) 1–12.
[2] R.E. Buck, R.W. Heath, P.L. Zenor, Strong monotone and nested normality, Topology Proc. 26 (1) (2001–2002) 67–82.
[3] J.G. Ceder, Some generalizations of metric spaces, Pacific J. Math. 11 (1961) 105–125.
[4] A. Dow, R. Ramírez, V.V. Tkachuk, A glance at spaces with closure-preserving lacal bases, Topology Appl. 157 (2010) 548–558.
[5] R. Engelking, General Topology, revised and completed edition, Heldermann Verlag, Berlin, 1989.
[6] G. Gruenhage, Generalized metric spaces, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, 1984, pp. 423–501.
[7] P.M. Gartside, Cardinal invariants of monotonically normal spaces, Topology Appl. 77 (1997) 303–314.
[8] R.W. Heath, P.L. Zenor, Stronger forms of monotone normality I, Abstract, in: Summer Conference on General Topology and Applications in Honor of Mary Ellen Rudin and Her Work, University of Wisconsin, Madison, WI, 1991.
[9] R.W. Heath, P.L. Zenor, Stronger forms of monotone normality II, Abstract, in: Summer Conference on General Topology and Applications in Honor of Mary Ellen Rudin and Her Work, University of Wisconsin, Madison, WI, 1991.
[10] M. Ito, $M_3$-spaces whose every point has a closure-preserving outer base are $M_1$, Topology Appl. 19 (1985) 65–69.
[11] Kuo-shih Kao, A note on $M_1$-spaces, Pacific J. Math. 108 (1983) 121–128.
[12] T. Mizokami, On closed subsets of $M_1$-spaces, Topology Appl. 141 (2004) 197–206.