Semi-quotient Mappings and Spaces With Compact-countable k-networks

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Abstract: In this paper the semi-quotient mappings are studied, and the concepts of wks-mappings and $wkcs$-mappings are introduced. It is shown that a space with a point-countable k-network if and only if it is a wks-image of a metric space, and a space with a compact-countable k-network if and only if it is a $wkcs$-image of a metric space, which answers a question posed by Chuan Liu and Y. Tanaka in 1996.

Key words: semi-quotient mappings; semi-pseudo-open mappings; ws-mappings; wks-mappings; $wkcs$-mappings; k-networks; wcs^*-networks

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0 Introuction

Spaces determined by point-countable covers are some important classes of generalized metric spaces[2]. A space has a point-countable base if and only if it is a continuous open s-image of a metric space. Every base for a space is a k-network, and every k-network is a wcs^*-network. It is a natural question how to characterize the spaces with a point-countable k-network or spaces with a point-countable wcs^*-network by a nice image of a metric space. In 1989, N. V. Velichko[10] introduced a semi-quotient ws-mapping, and proved that a sequential space has a point-countable k-network if and only if it is a semi-quotient ws-image of a metric space. In this paper we first analyze some properties of semi-quotient ws-mappings, introduce wk-mappings and wc-mappings, and characterize spaces with a point-countable k-network and spaces with a point-countable wcs^*-network, respectively, which improve N. V. Velichko's results. On the other hand, a space has a point-countable base if and only if it has a compact-countable base, but there is a space with a point-countable k-network which has no compact-countable k-network[8]. A question posed by Chuan Liu and Y. Tanaka[8] is still open as follows: Characterize Fréchet spaces with a compact-countable k-network by a nice image of a metric space. In this paper $wkcs$-mappings are defined and the question above is affirmatively answered.

In this paper all spaces are *Hausdorff*, and all mappings are onto. Readers may refer to [1]

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for unstated definitions.

1 Semi-quotient Mappings and Semi-pseudo-open Mappings

Definition 1[10] Suppose that a mapping $f: X \to Y$, and X_0 is a subspace of X. f is called continuous about X_0 if for each $x \in X$ and any neighborhood V of $f(x)$ in Y there is a neighborhood W of x in X such that $f(W \cap X_0) \subset V$.

Denote $f_0 = f_{|X_0} : X_0 \to Y$.

Lemma 1 Suppose that a mapping $f: X \to Y$ and $X_0 \subset X$. The following are equivalent:

(1) f is continuous about X_0.

(2) If a net $\{x_d\}_{d\in D}$ in X_0 converges to a point x in X, then a net $\{f(x_d)\}_{d\in D}$ converges to $f(x)$ in Y.

(3) If T is a subset of Y, then $\overline{f_0^{-1}(T)} \subset f^{-1}(\overline{T})$.

Proof (1) $\Rightarrow$ (2). Suppose that a mapping f is continuous about X_0, and a net $\{x_d\}_{d\in D}$ in X_0 converges to a point x in X. Let U be any neighborhood of $f(x)$ in Y, there is a neighborhood W of x in X such that $f(W \cap X_0) \subset U$, then there is a $d_0 \in D$ such that $x_d \in W \cap X_0$ for each $d \geq d_0$, thus $f(x_d) \in U$, hence the net $\{f(x_d)\}_{d\in D}$ converges to $f(x)$ in Y.

(2) $\Rightarrow$ (3). Suppose that $x \in \overline{f_0^{-1}(T)}$, there is a net $\{x_d\}_{d\in D}$ in $f_0^{-1}(T)$ converging to x in X, then $\{x_d\}_{d\in D} \subset X_0$ and $\{f(x_d)\}_{d\in D} \subset T$, so a net $\{f(x_d)\}_{d\in D}$ converges to $f(x) \in \overline{T}$ in Y, therefore $\overline{f_0^{-1}(T)} \subset f^{-1}(\overline{T})$.

(3) $\Rightarrow$ (1). Suppose that $\overline{f_0^{-1}(T)} \subset f^{-1}(\overline{T})$ for each $T \subset Y$. For each $x \in X$ and an open neighborhood V of $f(x)$ in Y, put $W = X \setminus \overline{f_0^{-1}(Y \setminus V)}$, since $x \notin f^{-1}(Y \setminus V)$ and $\overline{f_0^{-1}(Y \setminus V)} \subset f^{-1}(Y \setminus V)$, then W is an open neighborhood of x in X and $W \cap f_0^{-1}(Y \setminus V) = \emptyset$. This is $W \cap X_0 \cap f^{-1}(Y \setminus V) = \emptyset$, thus $f(W \cap X_0) \subset V$. So f is continuous about X_0.

By Lemma 1, the restriction $f_{|\overline{X}_0} : \overline{X}_0 \to Y$ is continuous $\Longrightarrow$ f is continuous about X_0 $\Longrightarrow$ the restriction $f_0 = f_{|X_0} : X_0 \to Y$ is continuous.

Lemma 2 Suppose that a mapping $f: X \to Y$ is continuous about X_0. If Y is a regular space, then the restriction $f_{|\overline{X}_0}$ is continuous.

Proof For each $x \in \overline{X}_0$ and any neighborhood V of $f(x)$ in Y, there is a neighborhood U of $f(x)$ in Y such that $f(x) \in U \subset \overline{U} \subset V$ by the regularity of Y. Since f is continuous about X_0, there is an open neighborhood W of x in X with $f(W \cap X_0) \subset U$. If $z \in W \cap \overline{X}_0$, there is a net $\{x_d\}_{d\in D}$ in X_0 converging to z in X, we can assume that each $x_d \in W$, then $f(x_d) \in U$ and the net $\{f(x_d)\}_{d\in D}$ converges to $f(z) \in \overline{U}$ by Lemma 1, thus $f(W \cap \overline{X}_0) \subset \overline{U} \subset V$. Hence $f_{|\overline{X}_0}$ is continuous.

Put $X = [0,1], Y = [0,1)$. X, Y are endowed with the usual Euclidean topology of subspaces in real line $\mathbb{R}$. Define a mapping $f: X \to Y$ by $f(0) = 0$ and $f(x) = 1 - x$ if $x \in (0,1]$, then the restriction $f_{|(0,1]} : (0,1] \to Y$ is continuous, but f is not continuous about (0, 1] by Lemma 2.

Definition 3[10] A mapping $f: (X, X_0) \to Y$ is called a semi-quotient ws-mapping if $X_0 \subset X$ and the following are satisfied:

(1) The restriction $f_0 = f_{|X_0} : X_0 \to Y$ is an s-mapping, i. e., $f_0^{-1}(y)$ is a separable subspace

of X_0 for each $y \in Y$.

(2) f is continuous about X_0.

(3) A subset T of Y is closed if and only if $\overline{f_0^{-1}(T)} \subset f^{-1}(T)$.

By Lemma 1, the condition (3) $\Rightarrow$ (2) in Definition 3. The semi-quotient ws-mappings are a composite concept. $f : (X, X_0) \to Y$ is called a ws-mapping if it satisfies the conditions (1) and (2); $f : (X, X_0) \to Y$ is called a semi-quotient mapping if it satisfies the condition (3) in Definition 3.

Lemma 4 Suppose that a mapping $f : (X, X_0) \to Y$ is continuous about X_0 and the restriction $f_{|X_0} : X_0 \to Y$ is a quotient mapping, then f is a semi-quotient mapping.

Proof Put $f_0 = f_{|X_0}$. If T is a subset of Y and $\overline{f_0^{-1}(T)} \subset f^{-1}(T)$, then $\mathrm{cl}_{X_0}(f_0^{-1}(T)) = \overline{f_0^{-1}(T)} \cap X_0 \subset f^{-1}(T) \cap X_0 = f_0^{-1}(T)$, thus $f_0^{-1}(T)$ is closed in X_0, hence T is closed in Y. By Lemma 1, f is a semi-quotient mapping.

By Lemma 1, if $f : (X, X_0) \to Y$ is semi-quotient, then f is continuous about X_0. It is well known that a mapping $f : X \to Y$ is pseudo-open if and only if $f(\overline{f^{-1}(T)}) = \overline{T}$ for each subset T of Y. Inspired by it, the following concept is defined.

Definition 5 Let $f : X \to Y$ be a mapping and X_0 a subspace of X. $f : (X, X_0) \to Y$ is called a semi-pseudo-open mapping if $f(\overline{f_0^{-1}(T)}) = \overline{T}$ for each subset T of Y.

Lemma 6 Suppose that a mapping $f : (X, X_0) \to Y$ is continuous about X_0 and the restriction $f_{|X_0} : X_0 \to Y$ is a pseudo-open mapping, then f is a semi-pseudo-open mapping.

Proof Let T be a subset of Y. $\overline{f_0^{-1}(T)} \subset f^{-1}(\overline{T})$ by Lemma 1, thus $f(\overline{f_0^{-1}(T)}) \subset \overline{T}$. Since $f_{|X_0} : X_0 \to Y$ is a pseudo-open mapping, $f_0(\mathrm{cl}_{X_0}(f_0^{-1}(T))) = \overline{T}$, thus $\overline{T} \subset f(\overline{f_0^{-1}(T)} \cap X_0) \subset f(\overline{f_0^{-1}(T)})$. Hence f is semi-pseudo-open.

Lemma 7 Every semi-pseudo-open mapping is semi-quotient.

Proof Let $f : (X, X_0) \to Y$ be a semi-pseudo-open mapping. If T is closed subset of Y, then $f(\overline{f_0^{-1}(T)}) = T$, thus $\overline{f_0^{-1}(T)} \subset f^{-1}(T)$. If a subset T of Y with $\overline{f_0^{-1}(T)} \subset f^{-1}(T)$, then $\overline{T} = f(\overline{f_0^{-1}(T)}) \subset T$, thus T is closed in Y. Hence f is semi-quotient.

2 Some Compact-covering Mappings

To characterize spaces with a point-countable k-network and spaces with a point-countable wcs^*-network as images of metric spaces under special mappings, we modify semi-quotient mappings as follows.

Definition 8 Suppose that a mapping $f : X \to Y$ is continuous about X_0.

(1) $f : (X, X_0) \to Y$ is called a wk-mapping if K is a compact subset of Y and T is a sequence in K, there is a sequence S in X_0 such that S has an accumulation in X and $f(S)$ is a subsequence of T.

(2) $f : (X, X_0) \to Y$ is called a wc-mapping if T is a convergent sequence in Y, there is a sequence S in X_0 such that S has an accumulation in X and $f(S)$ is a subsequence of T.

(3) $f : (X, X_0) \to Y$ is called a wks-mapping (wcs-mapping) if it is a wk-mapping (wc-mapping) and a ws-mapping.

Lemma 9 Every continuous closed mapping is a semi-quotient wks-mapping.

Proof Suppose that $f : X \to Y$ is a continuous closed mapping. For each $y \in Y$ take an $x_y \in f^{-1}(y)$, and put $X_0 = \{x_y : y \in Y\}$. Obviously, $f : (X, X_0) \to Y$ is continuous about X_0 and is a ws-mapping. If T is a subset of Y, and $\overline{f_0^{-1}(T)} \subset f^{-1}(T)$, then $\overline{T} = \overline{f(X_0 \cap f^{-1}(T))} = f(\overline{f_0^{-1}(T)}) \subset T$, thus T is closed in Y. Therefore f is a semi-quotient mapping. For a compact subset K of a space Y and an infinite sequence T in K, if the sequence $f_0^{-1}(T)$ of X_0 has no accumulation in X, then $f_0^{-1}(T)$ is a closed discrete subset of X, thus T is a closed discrete subset of K, a contradiction. So the sequence $f_0^{-1}(T)$ of X_0 has an accumulation in X. Thus every continuous closed mapping is a wks-mapping.

wk-mappings, wc-mappings are closely related to compact-covering mappings. Suppose that $f : X \to Y$ is a continuous mapping. f is called a compact-covering mapping[2] if K is a compact subset of Y, there is a compact subset L of X with $f(L) = K$; f is called a sequence-covering mapping[2] if T is a convergent sequence including the limit point in Y, there is a compact subset L in X with $f(L) = T$.

It is easy to check that

$$\begin{array}{ccccc} \text{compact-covering mappings} & \longrightarrow & wk\text{-mappings} & \longleftarrow & \text{continuous closed mappings} \\ \downarrow & & \downarrow & & \\ \text{sequence-covering mappings} & \longrightarrow & wc\text{-mappings.} & & \end{array}$$

A sequence-covering mapping may not be a wk-mapping. For example, let Y be the Stone-Čech compactification $\beta\mathbb{N}$, X be the set Y endowed with a discrete topology, and $f : X \to Y$ be an identical mapping. Since Y has no non-trivial convergent sequence[1], f is a sequence-covering mapping, but f is not a wk-mapping. On the other hand, a continuous closed mapping may not be a sequence-covering mapping. For example, let us consider a well-known Frolík's construction as follows. Put $\mathcal{D} = \{D \subset \mathbb{N} : D \text{ is infinite}\}$. For any $D \in \mathcal{D}$ choose $x_D \in \beta\mathbb{N} \setminus \mathbb{N}$ such that $x_D \in \mathrm{cl}(D)$. Then take $X = \mathbb{N} \cup \{x_D : D \in \mathcal{D}\} \subset \beta\mathbb{N}$. Finally define $f : X \to \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ by $f(X \setminus \mathbb{N}) = \{0\}$ and each $f(n) = \frac{1}{n}$ for each $n \in \mathbb{N}$. Then f is a continuous closed mapping of a regular space X in which every compact subset is finite onto the convergent sequence. Thus f is not a sequence-covering mapping.

Lemma 10 Let X be a Fréchet space. Then

(1) $f : (X, X_0) \to Y$ is a semi-quotient mapping if and only if f is a wk-mapping(wc-mapping) and Y is a sequential space.

(2) $f : (X, X_0) \to Y$ is a semi-pseudo-open mapping if and only if f is a wk-mapping(wc-mapping) and Y is a Fréchet space.

Proof (1) Suppose that $f : (X, X_0) \to Y$ is a semi-quotient mapping. Let K be a compact subset of Y and T be a sequence in K, then T has an accumulation z in K, we can assume that $z \notin T$, thus T is not closed in Y, there is $x \in \overline{f_0^{-1}(T)} \setminus f^{-1}(T)$. Since X is a Fréchet space, there is a sequence S in $f_0^{-1}(T)$ such that the sequence S converges to x. If there is a $y \in T$ such that $f_0^{-1}(y)$ contains a subsequence of S, then $x \in \overline{f_0^{-1}(y)} \subset f^{-1}(y)$, a contradiction with $f(x) \notin T$, hence we can consider that $f(S)$ is a subsequence of T. Therefore f is a wk-mapping.

Let F be a non-closed subset of Y. Then there is $x \in \overline{f_0^{-1}(F)} \setminus f^{-1}(F)$, and there is a sequence $\{x_n\}$ in $f_0^{-1}(F)$ which converges to x. Thus each $f(x_n) \in F$ and the sequence $\{f(x_n)\}$ converges to $f(x) \notin F$ by Lemma 1. Let $T = \{f(x_n) : n \in \mathbb{N}\} \cup \{f(x)\}$, then T is a convergent sequence in Y, and $T \cap F$ is not closed in Y. Therefore Y is a sequential space.

Conversely, let $f : (X, X_0) \to Y$ be a wc-mapping and Y a sequential space. If F is a non-closed subset of Y, there is a convergent sequence T including the limit point such that $T \cap F$ is not closed in Y, then the set $T \cap F$ is infinite, hence there is a sequence S in X_0 such that S has an accumulation x in X and $f(S)$ is a subsequence of $T \cap F$. Then $x \in \overline{f_0^{-1}(F)}$ and $f(x)$ is an accumulation of $f(S)$, so $f(x) \notin F$, i. e., $x \in \overline{f_0^{-1}(F)} \setminus f^{-1}(F)$, hence $\overline{f_0^{-1}(F)} \not\subset f^{-1}(F)$. Therefore $f : (X, X_0) \to Y$ is a semi-quotient mapping.

(2) Let $f : (X, X_0) \to Y$ be a semi-pseudo-open mapping. f is a wk-mapping by Lemma 7 and (1). If $y \in \overline{A} \subset Y$, there is a $x \in f^{-1}(y) \cap \overline{f_0^{-1}(A)}$ by $f(\overline{f_0^{-1}(A)}) = \overline{A}$, then there is a sequence $\{x_n\}$ in $f_0^{-1}(A)$ converging to x, thus the sequence $\{f(x_n)\}$ in A converging to y. Hence Y is a Fréchet space.

Conversely, let $f : (X, X_0) \to Y$ be a wc-mapping and Y a Fréchet space. For each subset F of Y, $f(\overline{f_0^{-1}(F)}) \subset \overline{F}$ by the continuity of f about X_0. On the other hand, let $y \in \overline{F}$, we need to show that $y \in f(\overline{f_0^{-1}(F)})$. There is a sequence T in F converging to y, thus there a sequence S in X_0 converging to x in X and $f(S)$ is a subsequence of T, then $x \in \overline{f_0^{-1}(F)}$ and $f(x)$ is an accumulation of $f(S)$, hence $y = f(x) \in f(\overline{f_0^{-1}(F)})$. Therefore $\overline{F} \subset f(\overline{f_0^{-1}(F)})$. So $f : (X, X_0) \to Y$ is a semi-pseudo-open mapping.

Question 11 Are sequential spaces preserved by semi-quotient mappings?

3 Spaces With a Point-countable wcs^*-network

Suppose that $\mathcal{P}$ is a family of subsets of a space X. $\mathcal{P}$ is called a k-network for X (see [2]) if for each compact subset K of X and any neighborhood U of K in X, there is a finite subset $\mathcal{F}$ of $\mathcal{P}$ with $K \subset \bigcup \mathcal{F} \subset U$. $\mathcal{P}$ is called a wcs^*-network for X (see [7]) if T is a sequence of X which converges to $x \in X$, and U is any neighborhood of x in X, there are a subsequence T' of T and a $P \in \mathcal{P}$ with $T' \subset P \subset U$.

Obviously, every k-network for a space X is a wcs^*-network.

Lemma 12[7] Let $\mathcal{P}$ be a point-countable family of subsets of a space X. Then $\mathcal{P}$ is a k-network for X if and only if $\mathcal{P}$ is a wcs^*-network for X and each compact subset of X is sequentially compact(or metrizable).

Thus a k-space having a point-countable k-network is a sequential space.

Lemma 13 Suppose that $\mathcal{B}$ is a point-countable base for a space X.

(1) If $f : (X, X_0) \to Y$ is a wks-mapping, then $f(\mathcal{B}_{|X_0})$ is a point-countable k-network for Y.

(2) If $f : (X, X_0) \to Y$ is a wcs-mapping, then $f(\mathcal{B}_{|X_0})$ is a point-countable wcs^*-network for Y.

Proof Denote $\mathcal{P} = f(\mathcal{B}_{|X_0})$. Since $\mathcal{B}_{|X_0}$ is a point-countable base for the subspace X_0, and $f_0 = f_{|X_0}$ is an s-mapping, then $\mathcal{P}$ is point-countable in Y.

(1) Suppose that $f : (X, X_0) \to Y$ is a wk-mapping. Let K be a compact subset of Y, and U be any neighborhood of K in Y, we shall show that there is a finite $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \bigcup\mathcal{F} \subset U$. If not, for each $y \in K$, denote $\{P \in \mathcal{P} : y \in P \subset U\} = \{P_i(y) : i \in \mathbb{N}\}$. There is a sequence $\{y_n\}$ of K such that $y_n \notin \bigcup\{P_i(y_k) : i, k < n\}$ for each $n \in \mathbb{N}$. Put $T = \{y_n : n \in \mathbb{N}\}$. Since f is a wk-mapping, there is a sequence $\{x_k\}$ of X_0 such that the sequence $\{x_k\}$ has an accumulation in X and $\{f(x_k)\}$ is a subsequence of $\{y_n\}$. Since X is a Fréchet space, we can assume that the sequence $\{x_k\}$ converges to a point x in X and $f(x) \notin T$. By Lemma 1, $f(x) \in \overline{T} \subset K$, thus U is a neighborhood of $f(x)$ in Y. And since f is continuous about X_0, there is a $B \in \mathcal{B}$ such that $x \in B$ and $f(B \cap X_0) \subset U$. There is a $y_{n_0} \in f(B \cap X_0)$ because the sequence $\{x_k\}$ converges to x and $\{f(x_k)\}$ is a subsequence of $\{y_n\}$, thus there is an $i_0 \in \mathbb{N}$ such that $f(B \cap X_0) = P_{i_0}(y_{n_0})$, so $y_n \notin P_{i_0}(y_{n_0})$ for each $n > \max\{i_0, n_0\}$. Hence the sequence $\{x_k\}$ has at most finite terms which do not belong to $f_0^{-1}(\{y_n : n \leq \max\{i_0, n_0\}\})$, there is an $n \leq \max\{i_0, n_0\}$ such that $\{x_k\}$ has infinite terms which is belonging to $f_0^{-1}(y_n)$, so $x \in \overline{f_0^{-1}(y_n)} \subset f^{-1}(y_n)$ by Lemma 1, thus $f(x) = y_n \in T$, a contradiction. Hence $\mathcal{P}$ is a point-countable k-network for Y.

(2) Suppose that $f : (X, X_0) \to Y$ is a wc-mapping. Let $\{y_n\}$ be a sequence in Y which converges to y and U be any neighborhood of y in Y, there is a sequence $\{x_i\}$ in X_0 such that the sequence $\{x_i\}$ has an accumulation in X and $\{f(x_i)\}$ is a subsequence of $\{y_n\}$. We can assume that the sequence $\{x_i\}$ converges to a point x in X. Thus $f(x) = y \in U$ by Lemma 1, there is a $B \in \mathcal{B}$ such that $x \in B$ and $f(B \cap X_0) \subset U$. Suppose that each $x_i \in B$, then each $f(x_i) \in f(B \cap X_0) \subset U$. So $\mathcal{P}$ is a point-countable wcs^*-network.

A mapping $f : X \to Y$ is called weakly continuous[3] if $f^{-1}(V) \subset [f^{-1}(\overline{V})]^\circ$ for each open set V in Y. $f : X \to Y$ is weakly continuous if and only if for each $x \in X$ and any neighborhood V of $f(x)$ in Y, there is a neighborhood W of x in X with $f(W) \subset \overline{V}$.

By the proof of Lemma 2, if f is continuous about X_0, then the restriction $f_{|\overline{X}_0}$ is weakly continuous. The converse proposition is not true. For example, take $X = [0, 2]$ endowed with the usual Euclidean topology, and $Y = [0, 1]$ endowed with a well-known Smirnov's deleted sequence topology as follows: Put $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, and V is open in Y if and only if there are a Euclidean open set G in [0, 1] and a $B \subset S$ such that $V = G \setminus B$. Then Y is a non-regular T_2-space. Define a mapping $f : X \to Y$ by $f(x) = x$ if $x \in [0, 1], f(x) = 2 - x$ if $x \in (1, 2]$. Thus f is weakly continuous, but f is not continuous about (0, 2].

The example above shows also the regularity of the space Y in Lemma 2 is essential. In fact, let $X_0 = X \setminus (\{0\} \cup \{1/n : n \in \mathbb{N}, n > 1\})$, then f is continuous about $X_0, \overline{X}_0 = X$, but f is not continuous.

A family $\mathcal{B}$ of subsets of a space X is called a π-network of a point x in X if V is any neighborhood of x in X, there is a $B \in \mathcal{B}$ such that $B \subset V$.

Lemma 14 (1) Spaces with a point-countable wcs^*-network are a weakly continuous wcs-image of a metric space.

(2) Spaces with a point-countable k-network are a weakly continuous wks-image of a metric space.

Proof (1) Suppose that $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is a point-countable wcs^*-network for a space X. Put $M = \{\alpha = (\alpha_i) \in A^\omega$: a family $\{U_{\alpha_i}\}_{i\in\mathbb{N}}$ has the finite intersection property and there is an $x_\alpha \in X$ such that $\{U_{\alpha_i}\}_{i\in\mathbb{N}}$ is a π-network of x_α in $X\}$. M is endowed with the subspace topology of the countably product space A^ω of a discrete space A, then M is a metric space. Define a mapping $f : M \to X$ by $f(\alpha) = x_\alpha$ for each $\alpha \in M$. Since X is a T_2-space, f is well-defined. Let $M_0 = \{\alpha = (\alpha_i) \in M : f(\alpha) \in U_{\alpha_i}$ for each $i \in \mathbb{N}\}$. We shall show that $f : (M, M_0) \to X$ is a weakly continuous wcs-mapping.

(a) Obviously, the restriction $f_0 = f_{|M_0} : M_0 \to X$ is an onto and s-mapping because $\mathcal{U}$ is a point-countable network for X.

(b) f is weakly continuous in M and continuous about M_0. Let $\alpha = (\alpha_i) \in M$ and V be any neighborhood of $f(\alpha)$ in X, then $U_{\alpha_k} \subset V$ for some $k \in \mathbb{N}$. Let $W = \{(\beta_i) \in M : \beta_k = \alpha_k\}$. Then W is an open neighborhood of α in M, $f(W \cap M_0) \subset V$ and $f(W) \subset \overline{V}$. In fact, if $\beta = (\beta_i) \in W \cap M_0$, then $\beta_k = \alpha_k$ and $f(\beta) \in \bigcap_{i\in\mathbb{N}} U_{\beta_i} \subset U_{\alpha_k} \subset V$. For each $\gamma = (\gamma_i) \in W$ and any neighborhood O of $f(\gamma)$ in X, $U_{\gamma_i} \subset O$ for some $i \in \mathbb{N}$, so $O \cap V \supset U_{\gamma_i} \cap U_{\alpha_k} = U_{\gamma_i} \cap U_{\gamma_k} \neq \emptyset$, hence $f(\gamma) \in \overline{V}$.

(c) f is a wc-mapping. If $T = \{x_n\}$ is a sequence in Y which converges to a point $x \notin T$. Put $T_k = \{x_n : n \geq k\}$ for each $k \in \mathbb{N}$. Let $\Phi = \{\mathcal{P}$: a finite subset $\mathcal{P}$ of $\mathcal{U}$ is a minimal cover of T_k for some $k \in \mathbb{N}\}$. By the point-countability of $\mathcal{U}$, Φ is countable, and denote $\Phi = \{\mathcal{P}_i : i \in \mathbb{N}\}$. Since $\{T_k\}_{k\in\mathbb{N}}$ has the finite intersection property, there is an ultrafilter $\mathcal{F}$ in the space X containing $\{T_k\}_{k\in\mathbb{N}}$. For each $i \in \mathbb{N}$, $\cup\mathcal{P}_i \in \mathcal{F}$, there is an $\alpha_i \in A$ such that $U_{\alpha_i} \in \mathcal{P}_i \cap \mathcal{F}$ because $\mathcal{F}$ is an ultrafilter in the space X. Then the family $\{U_{\alpha_i}\}_{i\in\mathbb{N}}$ has the finite intersection property. Suppose that V is any neighborhood of x in X, then $\{x\} \cup T_k \subset V$ for some $k \in \mathbb{N}$, so $\{x\} \cup T_k \subset \bigcup \mathcal{P}_j \subset V$ for some $j \in \mathbb{N}$. If not, for each $y \in \{x\} \cup T_k$, denote $\{U \in \mathcal{U} : y \in U \subset V\} = \{U_i(y) : i \in \mathbb{N}\}$. Then there is a sequence $\{y_n\}$ in $\{x\} \cup T_k$ such that $y_n \notin \bigcup\{U_i(y_j) : i, j < n\}$ for each $n \in \mathbb{N}$. Since the sequence $\{y_n\}$ converges to x, there is a $U \in \mathcal{U}$ such that $U \subset V$ and U contains infinite terms of the sequence $\{y_n\}$, a contradiction. So $U_{\alpha_j} \subset V$, then $\{U_{\alpha_i}\}_{i\in\mathbb{N}}$ is a π-network of x in X. Let $\alpha = (\alpha_i) \in A^\omega$, then $\alpha \in M$, $f(\alpha) = x$ and $\alpha \notin f^{-1}(T)$. For each $n \in \mathbb{N}$, put $B_n = \{(\beta_i) \in M : \beta_i = \alpha_i$ for each $i \leq n\}$, then $\{B_n : n \in \mathbb{N}\}$ is a local base of α in M, and $f(B_n \cap M_0) = \bigcap_{i\leq n} U_{\alpha_i}$ for each $n \in \mathbb{N}$. Since $\mathcal{F}$ is a filter in the space X, thus $T \cap (\cap_{i\leq n} U_{\alpha_i}) \neq \emptyset$, i. e., $T \cap f(B_n \cap M_0) \neq \emptyset$, then $B_n \cap f_0^{-1}(T) \neq \emptyset$, take a point $z_n \in B_n \cap f_0^{-1}(T)$. The sequence $\{z_n\}$ in M_0 converges to α in M, so there is a subsequence S of $\{z_n\}$ such that $f(S)$ is a subsequence of T.

In a word, f is a weakly continuous wcs-mapping.

(2) Suppose that $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is a point-countable k-network for a space X. It needs only to show that the mapping $f : (M, M_0) \to X$ defined above is a wk-mapping. For a compact subset K of X and any sequence $T = \{x_n\}$ in K, we can assume that the sequence $\{x_n\}$ converges to a point $x \in K \setminus \{x_n : n \in \mathbb{N}\}$ by Lemma 12. By the proof of (c) above, there

is a sequence S in M_0 such that S is convergent in M and $f(S)$ is a subsequence of T. Hence f is a wk-mapping.

We can show that M_0 is a dense subset of M in the proof of Lemma 14. In fact, for a non-empty open subset P in M, let $(\alpha_i) \in P$, there is an $n \in \mathbb{N}$ such that $B_n = \{(\beta_i) \in M : \beta_i = \alpha_i$ for each $i \le n\} \subset P$. Take an $x \in \cap_{i \le n} U_{\alpha_i}$, there is a $(\gamma_i) \in A^\omega$ such that $\{U_{\gamma_i}\}_{i \in \mathbb{N}}$ is a network of x in X and $\gamma_i = \alpha_i$ for each $i \le n$, then $(\gamma_i) \in P \cap M_0$. Hence $P \cap M_0 \ne \emptyset$, so M_0 is a dense subset of M. Thus the weak continuity of f in Lemma 14 can be obtained by the continuity about M_0 of f.

By Lemmas 13, 14 and 10, we have the following theorem and corollaries.

Theorem 15 (1) A space has a point-countable k-network if and only if it is a (weakly continuous) wks-image of a metric space.

(2) A space has a point-countable wcs^*-network if and only if it is a (weakly continuous) wcs-image of a metric space.

Corollary 16[10] A sequential space has a point-countable k-network if and only if it is a (weakly continuous) semi-quotient ws-image of a metric space.

If all spaces are regular, then the weakly continuous mapping can be enhanced a continuous mapping in Theorem 15 and Corollary 16.

Corollary 17[8,12] Every sequential space with a point-countable k-network is preserved by a continuous closed mapping.

Proof Suppose that $f : X \to Y$ is a continuous closed mapping, here X is a sequential space with a point-countable k-network. Y is a sequential space because sequential spaces are preserved by quotient mappings[6]. By Corollary 16, there is a metric space M and a weakly continuous semi-quotient ws-mapping $g : (M, M_0) \to X$. For each $y \in Y$, take an $x_y \in f^{-1}(y)$. Put $M_1 = g_0^{-1}(\{x_y : y \in Y\})$, $h = f \circ g : M \to Y$, and $h_1 = h_{|M_1}$. Since $M_1 \subset M_0$, $h : (M, M_1) \to Y$ is a ws-mapping and continuous about M_1. We shall show that h is a semi-quotient mapping. Suppose that T is a non-closed subset of Y, thus there is a sequence $\{y_n\}$ in T such that the sequence $\{y_n\}$ converges to $y \notin T$ in Y. For each $n \in \mathbb{N}$, put $x_n = x_{y_n}$, and let $S = \{x_n : n \in \mathbb{N}\}$. Since f is closed, S is not closed in X, and since X is a sequential space, the sequence $\{x_n\}$ has a convergent subsequence. We can assume that the sequence $\{x_n\}$ converges to a point x in X, then $f(x) = y$. Since g is semi-quotient and S is not closed in X, there is an $\alpha \in \overline{g_0^{-1}(S)} \setminus g^{-1}(S)$. If $g(\alpha) \ne x$, there is a neighborhood V of $g(\alpha)$ in X such that $\overline{V} \cap \overline{S} = \emptyset$, then there is a neighborhood W of α in M such that $g(W) \subset \overline{V}$ because g is weakly continuous, thus $W \cap g^{-1}(S) = \emptyset$, hence $\alpha \notin \overline{g_0^{-1}(S)}$, a contradiction. Thus $g(\alpha) = x$ and $h(\alpha) = y \notin T$. For each open neighborhood O of α in M, $O \cap h_1^{-1}(T) \supset O \cap g^{-1}(S) \cap M_1 = O \cap M_0 \cap g^{-1}(S) \ne \emptyset$, thus $\alpha \in \overline{h_1^{-1}(T)} \setminus h^{-1}(T)$, hence $\overline{h_1^{-1}(T)} \not\subset h^{-1}(T)$. Therefore h is semi-quotient. By Corollary 16, Y is a sequential space with a point-countable k-network.

Since a space with a point-countable k-network (wcs^*-network) is not preserved by a continuous closed mapping[11], a space with a point-countable k-network (wcs^*-network) is not preserved by a wks-mapping (wcs-mapping).

Example 18 The inverse proposition in Lemma 4 is not true. Let Y be the fan space S_{ω_1}, then Y is an image of a metric space X under a continuous closed mapping f (see [6]), there is a subspace X_0 of X such that $f:(X,X_0)\to Y$ is a semi-quotient ws-mapping by Lemma 9. Since S_{ω_1} is not any quotient s-image of a metric space[6], so $f_{|X_0}$ is not quotient.

Example 19 A continuous open mapping may not be a wcs-mapping. Let Y be the space ω_1 of countable order numbers with usual order topology. Since Y is a first countable space, there are a metric space X and a continuous open mapping $f:X\to Y$ (see [1]). Since Y has not any point-countable wcs^*-network by Lemma 12, f is not a wcs-mapping by Lemma 13.

4 Spaces With a Compact-countable k-network

A cs-mapping was introduced by Qu and Gao[9] to characterize spaces with a compact-countable k-network. Let $f:X\to Y$ be a mapping. f is called a cs-mapping if $f^{-1}(C)$ is separable for each compact subset C of Y. cs-images of metric spaces are discussed by some authors in [4, 5, 9].

Definition 20 A mapping $f:X\to Y$ is called a $wkcs$-mapping if $f:(X,X_0)\to Y$ is a wk-mapping and the restriction $f_0=f_{|X_0}:X_0\to Y$ is a cs-mapping.

Lemma 21 Suppose that $\mathcal{B}$ is a point-countable base for a space X. If $f:(X,X_0)\to Y$ is a $wkcs$-mapping, then $f(\mathcal{B}_{|X_0})$ is a compact-countable k-network for Y.

Proof Denote $\mathcal{P}=f(\mathcal{B}_{|X_0})$. Since $\mathcal{B}_{|X_0}$ is a point-countable base for the subspace X_0, and $f_0=f_{|X_0}$ is a cs-mapping, then $\mathcal{P}$ is compact-countable in Y. By Lemma 13, $\mathcal{P}$ is a compact-countable k-network for Y.

Lemma 22 Spaces with a compact-countable k-network are a weakly continuous $wkcs$-image of a metric space.

Proof Suppose that $\mathcal{U}=\{U_\alpha:\alpha\in A\}$ is a compact-countable k-network for a space X. It needs only to show that the mapping $f:(M,M_0)\to X$ defined in the proof of Lemma 14(1) is a $wkcs$-mapping. f is a wk-mapping by the proof of Lemma 14(2). Obviously, the restriction $f_0=f_{|M_0}:M_0\to X$ is an onto and cs-mapping because $\mathcal{U}$ is a compact-countable network for X.

In a word, f is a weakly continuous $wkcs$-mapping.

By Lemmas 21, 22 and 10, we have the following theorem.

Theorem 23 (1) A space has a compact-countable k-network if and only if it is a (weakly continuous) $wkcs$-image of a metric space.

(2) X is a k-space with a compact-countable k-network if and only if there are a metric space M and a (weakly continuous) semi-quotient mapping $f:(M,M_0)\to X$ such that $f_0:M_0\to X$ is a cs-mapping.

(3) X is a Fréchet space with a compact-countable k-network if and only if there are a metric space M and a (weakly continuous) semi-quotient mapping $f:(M,M_0)\to X$ such that $f_0:M_0\to X$ is a cs-mapping.

If all spaces are regular, then the weakly continuous mapping can be enhanced a continuous mapping in the above theorem.

Question 24 The following questions posed by Liu and Tanaka[8] are still open.

(1) Does every closed image of a space with a σ-locally countable k-network has a compact-countable k-network?

(2) Does every closed image of a k-space (or paracompact space) with a compact-countable k-network has a compact-countable k-network?

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半商映射与具有紧可数 k 网的空间

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摘要: 本文研究半商映射, 引进了 *wks* 映射和 *wkcs* 映射的概念, 证明了空间具有点可数 k 网当且仅当它是度量空间的 *wks* 映象, 空间具有紧可数 k 网当且仅当它是度量空间的 *wkcs* 映象, 这回答了刘川和田中祥雄 1996 年提出的问题.

关键词: 半商映射; 半伪开映射; *ws* 映射; *wks* 映射; *wkcs* 映射; k 网; wcs^* 网