Open Problems in Topology II – Edited by E. Pearl

Some problems on generalized metrizable spaces

Shou Lin

Generalized metrizable spaces are studied on the one hand to better understand the topology of metrizable spaces, and on the other to provide classes of non-metrizable spaces with some the desirable features of metrizable spaces. In the past years a number of excellent survey papers on generalized metrizable spaces have appeared. The paper [**12**] by Gruenhage was especially useful.

The theory of generalized metrizable spaces is closely related some questions about metrization theorems, mutual classifications of spaces and maps, countable product properties. Problems on generalized metrizable spaces are rich. In this chapter I shall pose only some problems about the images of metrizable spaces and connected spaces, and the spaces related hereditarily closure-preserving families. Some other problems about generalized metrizable spaces can be found, for examples, in G. Gruenhage's survey paper [**13**], and in G. Gruenhage's chapter, "Are stratifiable spaces M_1?" and C. Liu, Y. Tanaka's chapter, "Spaces and mappings, special networks" in this book.

All spaces are Hansdorff, and maps are continuous and onto. Readers may refer to [**9**] for unstated definitions and terminologies.

Sequence-covering maps

There are quite a few theorems about representing topological spaces as continuous images of spaces with additional properties. For examples, it is well-known that a space has a point-countable base if and only if it is an open and s-image of a metrizable space [**33**]. But, sometimes it is far from trivial to represent a space as an image of a metrizable space with some properties. Let $f\colon X \to Y$ be a map. f is called *sequence-covering* in the sense of Gruenhage, Michael and Tanaka [**14**] if in case S is a convergent sequence containing its limit point in Y then there is a compact subset K in X such that $f(K) = S$. Another definition about sequence-covering maps in the sense of Siwiec [**34**] is that $f\colon X \to Y$ is called *sequence-covering* if in case S is a convergent sequence in Y then there is a convergent sequence L in X such that $f(L) = S$, which is not used in this chapter. It was shown that every quotient and compact map of a metrizable space is sequence-covering [**24**], and every quotient and s-image of a metrizable space is a sequence-covering, quotient and s-image of a metrizable space [**14**]. Are those the best results? Let $f\colon (X,d) \to Y$ be a map with d a metric on X. f is a π-*map* with respect to d if for each $y \in Y$ and a neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$ [**33**]. Every compact map of a metric space is a π-map. There is a quotient and π-map f from a metric space onto a compact metric space in which f is not sequence-covering [**24**].

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1582? **Question 1.** *Is every quotient and π-image of a metric space also a sequence-covering and π-image of a metric space?*

A map $f\colon X \to Y$ is *compact-covering* [**25**] if in case C is a compact subset in Y then there is a compact subset K in X such that $f(K) = C$. Every compact-covering map is sequence-covering. There is a sequence-covering, quotient and compact map $f\colon X \to Y$ from a separable metric space X onto a compact metric space Y in which f is not compact-covering [**26**]. The following classic problem posed by Michael and Nagami in [**27**] has been answered negatively: Is every quotient s-image of a metric space a compact-covering, quotient s-image of a metrizable space. Chen in [**6**] gave a (sequence-covering,) quotient and compact image of a locally separable metrizable space which is not any quotient, compact-covering s-image of a metric space. And in [**7**] Chen constructed a regular example of a (sequence-covering,) quotient s-image of a metric space which is not any quotient, compact-covering s-image of a metric space under the assumption that there exists a σ'-set.

1583–1584? **Question 2.** *Let X be a regular space which is a (sequence-covering,) quotient and compact image of a metric space. Is X a compact-covering compact (resp. s-)image of a metrizable space?*

A map $f\colon X \to Y$ is called *bi-quotient* [**34**] if $f^{-1}(y)$ is covered by a family $\mathcal{U}$ consisting open subsets of X then there is a finite subset $\mathcal{U}'$ of $\mathcal{U}$ with $y \in \operatorname{int}(f(\bigcup\mathcal{U}'))$. Siwiec and Mancuso in [**35**] proved that a space Y is locally compact if and only if every compact-covering map onto Y is bi-quotient.

1585? **Question 3.** *Characterize the spaces Y such that every sequence-covering map onto Y is bi-quotient.*

A family $\mathcal{B}$ of subsets of a space X is called *point-regular* [**1**] if for every $x \in U$ with U open in X the set $\{B \in \mathcal{B} : x \in B \not\subset U\}$ is finite. It is a nice result that a space X is an open and compact image of a metrizable space if and only if X is a metacompact developable space [**15**], if and only if X has a point-regular base [**2**]. Let $\mathcal{P}$ be a family of subsets of a space X. $\mathcal{P}$ is called a *cs*-network* [**11**] for X if a sequence $\{x_n\}$ converges to a point $x \in U$ with U open in X, there exist $P \in \mathcal{P}$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x\} \cup \{x_{n_i} : i \in \mathbb{N}\} \subset P \subset U$. The following question was posed by Ikeda, Liu and Tanaka in [**16**]: For a sequential space X with a point-regular cs*-network, characterized X by means of a nice image of a metrizable space. It is easy to see that every sequence-covering and compact image of a metrizable space has a point-regular cs*-network. It was proved in [**41**] that a space X is a sequence-covering and compact image of a metrizable space if and only if X has a sequence $\{\mathcal{U}_n\}$ of point-finite covers satisfying that for each $n \in \mathbb{N}$ if S is a sequence converging to a point x in X then there are a $U_n \in \mathcal{U}_n$ for some $n \in \mathbb{N}$ and a subsequence L of S such that $\{x\} \cup L \subset U_n$.

1586? **Question 4.** *Is every space with a point-regular cs*-network a sequence-covering and compact image of a metrizable space?*

Connectedness is less closely related the properties of generalized metrizable spaces. A space is called *sequentially connected* if it cannot be expressed as the union of two non-empty disjoint sequentially open subsets [**10**]. Every connected and sequential space is sequentially connected, and every sequentially connected space is connected. Recently, it was shown in [**23**] that a space is sequentially connected if and only if it is a sequence-covering image of a connected metrizable space. Thus every connected and sequential (resp. Fréchet–Urysohn) space is a quotient (resp. pseudo-open) image of a connected metrizable space [**10**, **23**]. It is known that a space is a k-space (resp. a first-countable space) if and only if it is a quotient (resp. an open) image of a paracompact locally compact (resp. a metrizable) space.

Question 5. *Are k-and connected spaces the quotient images of connected paracompact locally compact spaces?* 1587?

Question 6. *Are first-countable connected spaces the open images of connected metrizable spaces?* 1588?

σ-spaces and Σ-spaces

Let us recall some related generalized metrizable spaces. Let X be a topological space and $\mathcal{P}$ a cover of X. $\mathcal{P}$ is called a *quasi-(modk)-network* (resp. *(modk)-network*) [**32**] for X if there is a closed cover $\mathcal{K}$ by countably compact(resp. compact) subsets of X such that, whenever $K \in \mathcal{K}$ and $K \subset U$ with U open in X, then $K \subset P \subset U$ for some $P \in \mathcal{P}$. $\mathcal{P}$ is called a *network* for X if $\mathcal{P}$ is a (mod$k$)-network with $\mathcal{K} = \{\{x\} : x \in X\}$.

According to the Bing–Nagata–Smirnov metrization theorem, some generalized metrizable spaces were introduced. A space X is called a *σ-space* [**31**] if it is a regular space with a σ-locally finite network. A space X is called a *Σ-space* [**32**] (resp. a *strong Σ-space* [**30**]) if it has a σ-locally finite quasi-(mod$k$)-network (resp. (mod$k$)-network) by closed subsets. A space X is called *semi-stratifiable* [**8**] if, for each open set U of X, one can assign a sequence $\{F(n,U)\}_{n\in\mathbb{N}}$ of closed subsets of X such that

(1) $U = \bigcup_{n\in\mathbb{N}} F(n,U)$;
(2) $F(n,U) \subset F(n,V)$ whenever $U \subset V$.

Lašnev [**18**] proved that if X is metrizable and $f\colon X \to Y$ is a closed map, then $f^{-1}(y)$ is compact for all $y \in Y$ outside of some σ-closed discrete subset of Y. Some extensions of Lašnev's theorem to, e.g., σ-spaces [**5**], normal semi-stratifiable spaces [**36**], perfect pre-images of normal σ-spaces, are known to hold.

Question 7. *Is $f^{-1}(y)$ compact for all $y \in Y$ outside of some σ-closed discrete subset of Y if X is a perfect pre-image of a normal semi-stratifiable space and $f\colon X \to Y$ is a closed map?* 1589?

A family $\mathcal{P}$ of subsets of a space X is called *hereditarily closure-preserving* [**19**] if the family $\{H(P) : P \in \mathcal{P}\}$ is closure-preserving for each $H(P) \subset P \in \mathcal{P}$, i.e., $\overline{\bigcup\{H(P) : P \in \mathcal{P}'\}} = \bigcup\{\overline{H(P)} : P \in \mathcal{P}'\}$ for each $\mathcal{P}' \subset \mathcal{P}$. A space X is called a

Σ^**-space* (resp. a *strong* Σ^**-space*) [**32**] if it has a σ-hereditarily closure-preserving quasi-(mod$k$)-network (resp. (mod$k$)-network) by closed subsets. A regular space is a σ-space if and only if it is a semi-stratifiable and Σ^*-space [**17**]. Tanaka and Yajima in [**39**] proved the following a theorem for Σ-spaces. If X is a Σ-space and $f\colon X \to Y$ is a closed map, then $f^{-1}(y)$ is $\aleph_1$-compact for all $y \in Y$ outside of some σ-closed discrete subset of Y. In [**21**] the author tried to obtain a similar result as mentioned above for Σ^*-spaces, but its proof has a gap.

1590? **Question 8.** *Is $f^{-1}(y)$ $\aleph_1$-compact for all $y \in Y$ outside of some σ-closed discrete subset of Y if X is a Σ^*-space and $f\colon X \to Y$ is a closed map?*

It is a classic and important result that a regular space is a σ-space if and only if it has a σ-discrete network. Buhagiar and Lin in [**3**] showed that a space X is a strong Σ-space if and only if it has a σ-discrete (mod$k$)-network by closed subsets.

1591? **Question 9.** *Does every Σ-space have a σ-discrete quasi-(modk)-network by closed subsets?*

As for the product property of Σ-spaces, Okuyama in [**32**] proved that a space X is a Σ-space if and only if $X \times [0,1]$ is a Σ^*-space for a paracompact space X. It is known that a space X is a strong Σ-space if and only if it is a subparacompact Σ-space [**3**].

1592? **Question 10.** *Is X a strong Σ-space if $X \times [0,1]$ is a strong Σ^*-space?*

$\aleph_0$-spaces

A family $\mathcal{P}$ of subsets of a space X is called a *pseudo-base* if $\mathcal{P}$ is a (mod$k$)-network with $\mathcal{K} = \{K : K \text{ is compact in } X\}$. A space X is called an $\aleph_0$*-space* [**25**] if it is a regular space with a countable pseudo-base. It is easy to check that $\aleph_0$-spaces are preserved by closed maps. However, the regular image of an $\aleph_0$-space under an open map cannot be an $\aleph_0$-space [**25**].

1593? **Question 11.** *Is the regular image of an $\aleph_0$-space under an open and compact map an $\aleph_0$-space?*

Spaces related to pseudo-bases are special. For example, Lin in [**20**] obtained that a regular space is an $\aleph_0$-space if and only if it has a point-countable pseudo-base, and a regular space has a σ-hereditarily closure-preserving pseudo-base if and only if either it is an $\aleph_0$-spaces or it is a σ-closed discrete space in which all compact subsets are finite. On the other hand, some generalizations of the families about compact-finite families or hereditarily closure-preserving families were introduced by T. Mizokami in [**28**] as follows. A family $\mathcal{P}$ of subsets of a space X is called *CF* in X if $\mathcal{P}_{|K} = \{P \cap K : P \in \mathcal{P}\}$ is finite for each compact subset K of X, and called CF^* in X if additionally the family $\{P \in \mathcal{P} : P \cap K = P'\}$ is finite for each infinite subset $P' \in \mathcal{P}_{|K}$. It is easy to check that

Hereditarily closure-preserving family $\Rightarrow$ CF^* family $\Rightarrow$ CF family.

It was shown in [**29**] that a regular space X has a σ-CF^* pseudo-base consisting of perfect subsets of X if and only if X is either an $\aleph_0$-space or a space in which all compact subsets are finite.

Question 12. *Let X be a regular space with a σ-CF^* pseudo-base. Is X either an $\aleph_0$-space or a space in which all compact subsets are finite?* 1594?

A family $\mathcal{P}$ of subsets of a space X is called a *quasi-base* [**4**] if, whenever $x \in X$ and U is a neighborhood of x in X, then there exists a $P \in \mathcal{P}$ such that $x \in \operatorname{int}(P) \subset P \subset U$. A regular space is metrizable if and only if it has a σ-compact finite base [**22**], if and only if it is a k-space with a σ-CF base [**28**], if and only if it is a k-space with a σ-CF^* quasi-base [**42**], if and only if it is a k'-space with a σ-CF quasi-base [**40**].

Question 13. *Let X be a regular space. Is X metrizable if it is a k-space with a σ-CF quasi-base?* 1595?

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