On countable-to-one maps

Chuan Liu a,*, Shou Lin b

a Department of Mathematics, Ohio University-Zanesville, Ohio, OH 43701, USA
b Institute of Mathematics, Ningde Teachers’ College, Fujian 352100, PR China

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Abstract

In this paper, it is proved that a space with a point-countable base is an open, countable-to-one image of a metric space, and a quotient, countable-to-one image of a metric space is characterized by a point-countable $\aleph_0$-weak base. Examples are provided in order to answer negatively questions posed by Gruenhage et al. [G. Gruenhage, E. Michael, Y. Tanaka, Spaces determined by point-countable covers, Pacific J. Math. 113 (1984) 303–332] and Tanaka [Y. Tanaka, Closed maps and symmetric spaces, Questions Answers Gen. Topology 11 (1993) 215–233].

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1. Introduction

The certain images of metric spaces have been studied extensively in the past years [6]. It is well known that a $T_0$-space has a point-countable base if and only if it is an open $s$-image of a metric space [3], here $f : X \to Y$ is an $s$-map if each fiber $f^{-1}(y)$ is separable in $X$. G. Gruenhage et al. [4] showed that spaces determined by point-countable covers are preserved by quotient maps with countable fibers. Every countable-to-one map is an $s$-map. Are quotient countable-to-one images on metric spaces and quotient $s$-images on metric spaces coincident? The question is discussed and some related results are obtained in this paper.

Throughout this paper, all spaces are assumed to be $T_2$, all maps are continuous and onto. Denote real, irrational and rational numbers by $\mathbb{R}$, $\mathbb{P}$ and $\mathbb{Q}$, respectively. We refer the reader to [2] for notations and terminology not explicitly given here.

2. Main results

Theorem 1. The following are equivalent for a space $X$:

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* Corresponding author. Tel.: +1 740 588 1419; fax: +1 740 453 6161.
E-mail addresses: liuc1@ohio.edu (C. Liu), linshou@public.ndptt.fj.cn (S. Lin).

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Lemma 4. It is well known that (1) and (2) are equivalent. (3) ⇒ (2) is obvious. We prove that (2) ⇒ (3).

Let \( f : M \to X \) be an open s-map from a metric space \( M \) onto the space \( X \). For each \( x \in X \), let \( D_x \) denote a countable dense subset of \( f^{-1}(x) \) because \( f^{-1}(x) \) is separable. Put \( D = \bigcup \{D_x : x \in X\} \), and \( g = f|_D : D \to X \). Then \( g \) is a countable-to-one map. We prove that \( g \) is open. Let \( U \) be an open subset of \( D \). There is an open subset \( V \) in \( M \) such that \( U = V \cap D \). If \( g(U) \) is not open in \( X \), there is \( x \in g(U) \cap X \setminus g(U) \). Since \( X \) is first countable, there is a sequence \( \{x_n\} \) in \( X \setminus g(U) \) with \( x_n \to x \) in \( X \). Because \( x \in f(V) \) and \( f(V) \) is open in \( X \), without loss of generality, we can assume that each \( x_n \in f(V) \). Thus \( f^{-1}(x_n) \cap V \neq \emptyset \) and \( D_{x_n} \cap V \neq \emptyset \). Pick \( y_n \in D_{x_n} \cap V \subset U \), then \( x_n = g(y_n) \in g(U) \), a contradiction. Thus \( g(U) \) is open in \( X \). Hence \( g \) is an open map and \( X \) is an open, countable-to-one image of the metric space \( D \). \( \square \)

Definition 2. Let \( B \) be a family of subsets of a space \( X \). \( B \) is said to be an \( \aleph_0 \)-weak base for \( X \) if \( B = \bigcup \{B_x(n) : x \in X, n \in \mathbb{N}\} \) satisfies

1. For each \( x \in X \), \( n \in \mathbb{N} \), \( B_x(n) \) is closed under finite intersections and \( x \in \bigcap B_x(n) \).
2. A subset \( U \) of \( X \) is open if and only if whenever \( x \in U \) and \( n \in \mathbb{N} \), there exists a \( B_x(n) \in B_x(n) \) such that
   \[ B_x(n) \subset U. \]

\( X \) is called \( \aleph_0 \)-weakly first-countable [10] or weakly quasi-first-countable in the sense of Sirois-Dumais [9] if \( B_x(n) \) is countable for each \( x \in X, n \in \mathbb{N} \).

If \( B_x(n) = B_x(1) \) for each \( n \in \mathbb{N} \) in the definition of \( \aleph_0 \)-weak bases, the \( B \) is said to be a weak base for \( X \) [1]. \( X \) is called weakly first-countable or \( g \)-first countable in the sense of Arhangel’skii [1] if \( B_x(1) \) is countable for each \( x \in X \).

Let \( X \) be a space. \( P \subset X \) is called a sequential neighborhood of \( x \) in \( X \), if each sequence converging to \( x \) in \( X \) is eventually in \( P \). A subset \( U \) of \( X \) is called sequentially open if \( U \) is a sequential neighborhood of each of its points. \( X \) is called a sequential space if each sequential open subset of \( X \) is open.

Lemma 3. [9] Every \( \aleph_0 \)-weakly first-countable space is sequential.

Let \( f : X \to Y \) be a map. \( f \) is called subsequence-covering if whenever \( L \) is a convergent sequence in \( Y \) there is a convergent sequence \( S \) in \( X \) such that \( f(S) \) is a subsequence of \( L \).

Lemma 4. [6] Let \( f : X \to Y \) be a map, and \( Y \) a sequential space. Then \( f \) is quotient if and only if \( Y \) is a sequential space and \( f \) is subsequence-covering.

Theorem 5. \( X \) is a quotient, countable-to-one image of a metric space if and only if \( X \) has a point-countable \( \aleph_0 \)-weak base.

Proof. Necessity. Let \( f : M \to X \) be a quotient, countable-to-one map from a metric space \( M \) onto the space \( X \). Let \( B \) be a point-countable base for \( M \). For each \( y \in M \), let \( B_y \subset B \) be a countable, decreasing local base at \( y \) in \( M \). Put \( B' = \{B_y : y \in M\} \). Then \( B' \) is a point-countable family of \( M \). Since \( f \) is a countable-to-one map, \( f(B') \) is point-countable in \( X \). We shall check that \( f(B') \) is an \( \aleph_0 \)-weak base for \( X \).

For each \( y \in M \), denote \( B_y \) by \( \{B_{y,i} : i \in \mathbb{N}\} \) with each \( B_{y,i+1} \subset B_{y,i} \). For each \( x \in X \), denote \( f^{-1}(x) \) by \( \{x_n : n \in \mathbb{N}\} \). Let \( \mathcal{P}_x(n) = f(B_{x,n}) \). Then \( f(B') = \bigcup \{\mathcal{P}_x(n) : x \in X, n \in \mathbb{N}\} \). Let \( U \) be open in \( X \). For each \( x \in U, n \in \mathbb{N}, x_n \in f^{-1}(U) \), \( B_{x,n,i} \subset f^{-1}(U) \) for some \( i \in \mathbb{N} \), thus \( f(B_{x,n,i}) \subset \mathcal{P}_x(n) \) and \( f(B_{x,n,i}) \subset U \). On the other hand, let \( U \) be a subset of \( X \) satisfying for each \( x \in U, n \in \mathbb{N} \), there exist \( i \in \mathbb{N} \) such that \( f(B_{x,n,i}) \subset U \). We prove that \( U \) is open in \( X \). Since \( f \) is quotient, \( X \) is a sequential space by Lemma 4, it suffices to prove that \( U \) is sequential open in \( X \). Suppose that \( U \) is not sequential open, there is a sequence \( L \) in \( X \setminus U \) converging to \( x \in U \). Since \( f \) is a quotient
map, there is a sequence $S$ converging to some $x_0 \in f^{-1}(x)$ in $M$ such that $f(S)$ is a subsequence of $L$ by Lemma 4. Since the sequence $S$ is eventually in $B_{x_0,i}$, thus the sequence $f(S)$ is eventually in $f(B_{x_0,i}) \subset U$, a contradiction. Thus $U$ is sequential open. Hence, $X$ has a point-countable $\aleph_0$-weak base.

Sufficiency. Let $B = \bigcup \{B_x(n): x \in X, \ n \in \mathbb{N}\}$ be a point-countable $\aleph_0$-weak base, here each $B_x(n) = \{B_x(n, m): m \in \mathbb{N}\}$ with $B_x(n, m + 1) \subset B_x(n, m)$ for each $m \in \mathbb{N}$. Then any subsequence $B_x'$ of $(B_x(n, m))_{m \in \mathbb{N}}$ is a network at $x$ in $X$ for each $x \in X$ and $n \in \mathbb{N}$, i.e., if $U$ is an open neighborhood of $x$ in $X$, then $x \in B \subset U$ for some $B \in B_x'$. We rewrite $B = \{B_x: \alpha \in I\}$. Endow $I$ with discrete topology and let $I_x$ be a copy of $I$ for each $x \in X$. For convenience’ sake, two families $\{P_{n\in\mathbb{N}}\}$ and $\{Q_{n\in\mathbb{N}}\}$ of subsets of a space are said to be cofinal if there exist $n_0, m_0 \in \mathbb{N}$ such that $P_{n_0-i} = Q_{m_0+i}$ for every $i \in \mathbb{N}$. Put

$$M = \left\{ \alpha = (\alpha_i) \in \prod_{Ii \in \mathbb{N}} I_i: \{B_{x_i}\}_{i \in \mathbb{N}} \text{ is cofinal to } B_{\alpha_i}(n) \text{ for some } x_\alpha \in X, \ n \in \mathbb{N}, \ \{B_{x_i}\}_{i \in \mathbb{N}} \text{ is a network of } x_\alpha \right\}. $$

Define $f: M \rightarrow X$ as $f(\alpha) = x_\alpha$. It is easy to see that $f$ is well-defined and onto because $X$ is Hausdorff and each $B_x(n)$ is a network of $x$ in $X$ for each $n \in \mathbb{N}$. And $f(\alpha) = \bigcap_{i \in \mathbb{N}} B_{\alpha_i}$ for each $\alpha = (\alpha_i) \in M$. Notice that $B$ is point-countable, then $f$ is countable-to-one. Also $f$ is continuous, in fact, for any neighborhood $U$ of $x_\alpha$, there exists $m \in \mathbb{N}$ such that $B_{x_{\alpha_m}} \subset U$. Let $V = (I_1 \times \cdots \times \{t_m\} \times I_{m+1} \times \cdots) \cap M$, then $V$ is an open neighborhood of $\alpha$ in $M$ and $f(V) \subset B_{x_{\alpha_m}} \subset U$, hence $f$ is continuous.

To prove that $f$ is a quotient map, we only need to prove that $f$ is a subsequence-covering map by Lemmas 3 and 4.

**Claim.** Let $L$ be a sequence converging to $x \notin L$ in $X$. Then there exist a subsequence $L'$ of $L$ and $n_0 \in \mathbb{N}$ such that $L'$ is eventually in $B_x(n_0, m)$ for any $m \in \mathbb{N}$.

In fact, since the set $L$ is not closed in $X$, there is $n_0 \in \mathbb{N}$ such that $B_x(n_0, m) \cap L \neq \emptyset$ for any $m \in \mathbb{N}$ by Definition 2. If $B_x(n_0, m) \cap L$ is finite for some $m \in \mathbb{N}$, then $B_x(n_0, m) \subset X \setminus (B_x(n_0, m) \cap L)$ for some $k \geq m$, thus $B_x(n_0, k) \cap L = \emptyset$, a contradiction. So $B_x(n_0, m) \cap L$ is infinite for any $m \in \mathbb{N}$, hence there exist a subsequence $L'$ of $L$ such that $L'$ is eventually in $B_x(n_0, m)$ for any $m \in \mathbb{N}$.

For each $i \in \mathbb{N}$, take $\alpha_i \in I_i$ with $B_{\alpha_i} = B_x(n_0, i)$. Let $\alpha = (\alpha_i)$, then $\alpha \in M$. For each $k \in \mathbb{N}$, put $n_k = \min\{m \in \mathbb{N}: x_k \notin B_x(n_0, m)\}$. Construct $z_k = (\beta_i(k)) \in \prod_{Ii \in \mathbb{N}} I_i$ as follows: if $i < n_k$, pick $\beta_i(k) \in I_i$ with $B_{\beta_i(k)} = B_x(n_0, i)$; otherwise pick $\beta_i(k) \in I_i$ such that $B_{\beta_i(k)} = B_x(1, i - n_k + 1)$. Then $\{B_{\beta_i(k)}\}_{i \in \mathbb{N}}$ is cofinal to $B_x(1, i)$, thus $z_k \in M$ and $f(z_k) = x_k$. On the other hand, for each $i \in \mathbb{N}$, there is $k_0 \in \mathbb{N}$ such that $x_k \in B_x(n_0, i)$ for any $k \geq k_0$ because $L'$ is eventually in $B_x(n_0, i)$. Then $i < n_k$ when $k \geq k_0$ by the definition of $n_k$, so $\beta_i(k) = \alpha_i$. It implies that the sequence $\{\beta_i(k)\}_{i \in \mathbb{N}}$ converges to $\alpha_i$ in the discrete space $I_i$. Hence, $\{z_k\}$ converges to $\alpha$ in $M$. Therefore, $f$ is subsequence-covering, and $f$ is a quotient map. □

It is natural to ask whether a quotient $\sigma$-image of a metric space is a quotient, countable-to-one image of a metric space. The following example shows that the answer is “no”.

**Example 6.** There is a closed image of a separable metric space, which is not $\aleph_0$-weakly first-countable.

**Proof.** Let $X = \mathbb{R}^2 \setminus (\mathbb{Q} \times \{0\})$ be endowed with the subspace topology of $\mathbb{R}^2$ with the usual topology. Then $X$ is a separable metric space. Let $Y$ be the quotient space from $X$ by identifying $\mathbb{P} \times \{0\}$ to a point. It is obvious that the quotient map is a closed map. It has been proved that if an image of a metric space under a closed map is $\aleph_0$-weakly first-countable, then the each boundary of fibers is $\sigma$-compact by Theorem 2.1 in [7]. Since $\mathbb{P} \times \{0\}$ is not $\sigma$-compact in $X$, $Y$ is not $\aleph_0$-weakly first-countable. □

We do not know if a quotient, $\sigma$-compact image of a metric space is a quotient, countable-to-one image of a metric space. We shall give a partial answer to the question.

Recall some related concepts. Let $X$ be a space. A family $\mathcal{P}$ of subsets of $X$ is said to be a cs-network [5] for $X$, if whenever $U$ is an open set and a sequence $\{x_n\}$ in $X$ converges to a point in $U$, then $\{x_n\}$ is eventually in $P$ and $P \subset U$ for some $P \in \mathcal{P}$. A space is said to be an $\aleph_0$-space [5], if it has a countable cs-network.
Theorem 7. The following are equivalent for a space $X$:

1. $X$ is a quotient, countable-to-one image of a separable metric space.
2. $X$ is a quotient, $\sigma$-compact image of a separable metric space.
3. $X$ is $\aleph_0$-weakly first-countable and a quotient image of a separable metric space.
4. $X$ has a countable $\aleph_0$-weak base.
5. $X$ is an $\aleph_0$-weakly first-countable and $\aleph_0$-space.

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3) due to [9]. (3) $\Rightarrow$ (5) is obvious [3]. We shall prove that (5) $\Rightarrow$ (4) $\Rightarrow$ (1). Let $\mathcal{P}$ be a countable $cs$-network which is closed under finite intersections. Let $\bigcup\{B_x(n): x \in X, n \in \mathbb{N}\}$ be an $\aleph_0$-weak base for $X$, here each $B_x(n) = \{B_x(n, m): m \in \mathbb{N}\}$ with $B_x(n, m + 1) \subset B_x(n, m)$ for each $m \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\mathcal{P}_x(n) = \{P \in \mathcal{P}: B_x(n, m) \subset P \text{ for some } m \in \mathbb{N}\}$. Then $\mathcal{P}_x(n)$ is closed under finite intersections.

$\mathcal{P}_x(n)$ is a network of $x$ in $X$. In fact, suppose not, there is a neighborhood $U$ of $x$ in $X$ such that $P \not\subset U$ for each $P \in \mathcal{P}_x(n)$. Put $\{P \in \mathcal{P}: x \in P \subset U\} = \{P_k: k \in \mathbb{N}\}$. Then $B_x(n, m) \not\subset P_k$ for any $m, k \in \mathbb{N}$. Pick $x_{mk} \in B_x(n, m) \setminus P_k$ for each $m \geq k$. Let $y_i = x_{mk}$, where $i = k + m(m - 1)/2$. Then the sequence $\{y_i\}$ converges to $x$ in $X$ because $B_x(n, m)_{m \in \mathbb{N}}$ is a decreasing network of $x$ in $X$. Since $\mathcal{P}$ is a $cs$-network for $X$, there exist $k, j \in \mathbb{N}$ such that $y_i: i \geq j \in \mathcal{P}_k$. Pick $i \geq j$ such that $y_i = x_{mk}$ for some $m \geq k$, then $x_{mk} \in P_k$, a contradiction.

Put $\mathcal{B} = \bigcup\{\mathcal{P}_x(n): x \in X, n \in \mathbb{N}\}$. Then $\mathcal{B}$ is countable. We shall prove that $\mathcal{B}$ is an $\aleph_0$-weak base for $X$. We only need to prove that a subset $V$ of $X$ is open if whenever $x \in V$, $n \in \mathbb{N}$, there exists a $P_x(n) \in \mathcal{P}_x(n)$ such that $P_x(n) \subset V$. If $V$ is not open in $X$, then $V$ is not sequentially open because $X$ is sequentially compact. There is a sequence $L \subset X$, $x \in X$ converging to $x$, by the claim in the proof of Theorem 5, there exist $L' \subset L$ and $n_0 \in \mathbb{N}$ such that $L'$ is sequentially open in $B_x(n_0, m)$ for any $m \in \mathbb{N}$. But $B_x(n_0, m) \cap \mathcal{P}_x(n_0)$ for some $m \in \mathbb{N}$, so $L'$ is sequentially open in $B_x(n_0) \subset V$, a contradiction. Hence, $\mathcal{B}$ is a countable $\aleph_0$-weak base for $X$.

(4) $\Rightarrow$ (1) similar to the proof of the Sufficiency of Theorem 5, where each $I_i$ is countable and $M$ is a separable metric space. $\square$

In the final part of this section we discuss the closed, countable-to-one images of metric spaces. A space $X$ is said to be a Fréchet space if whenever $x \in \overline{A}$ in $X$ there is a sequence in $A$ which converges to $x$ in $X$. A space $X$ is determined by a cover $\mathcal{P}$ if $U \subset X$ is open (closed) in $X$ if and only if $U \cap P$ is open (closed) in $P$ for each $P \in \mathcal{P}$ [4].

Theorem 8. Let $X$ be a Fréchet space determined by a countable cover of closed metric subsets. Then $X$ is a closed, countable-to-one image of a metric space.

Proof. Suppose that $X$ is determined by a countable cover $\{X_n\}_{n \in \mathbb{N}}$ of closed metric subsets. Let $Y_n = X_n \setminus \bigcup\{X_i: i < n\}$, $Z_n = Y_n$ for each $n \in \mathbb{N}$. Then $Y_i \cap Y_j = \emptyset$ if $i \neq j$. Note that if $x_n \in Y_n$, $\{x_n: n \in \mathbb{N}\}$ is a closed discrete subspace of $X$. In fact, if $A \subset \{x_n: n \in \mathbb{N}\}$, then $A \cap X_n \subset \{x_i: i \leq n\}$, which is closed in $X_n$ for each $n \in \mathbb{N}$. Thus $A$ is closed in $X$ because $X$ is determined by $\{X_n: n \in \mathbb{N}\}$.

Let $f: \bigoplus_{n \in \mathbb{N}} Z_n \to X$ be the obvious map. Then $f$ is a countable-to-one map. Let $A$ be a closed subset in $\bigoplus_{n \in \mathbb{N}} Z_n$.

Claim. $f(A)$ is closed in $X$.

Suppose not, there is a sequence $\{y_n\}$ in $f(A)$ with $y_n \to y \notin f(A)$. If $A \cap Z_{i_0} \cap \{y_n: n \in \mathbb{N}\}$ is infinite for some $i_0 \in \mathbb{N}$, $y \in A \cap Z_{i_0}$ as $A \cap Z_{i_0}$ is closed. Thus $y \in f(A)$, a contradiction. Hence, $A \cap Z_i \cap \{y_n: n \in \mathbb{N}\}$ is infinite for each $i \in \mathbb{N}$. There is a subsequence $\{z_k\}$ of $\{y_n\}$ such that $z_k \in A \cap Z_{i_k}$ with each $i_k < i_{k+1}$. For each $k \in \mathbb{N}$, there is a sequence $\{x_n(k)\}$ in $Y_{i_k}$ with $x_n(k) \to z_k$ in $X$. Thus $y \in \{x_n(k): n, k \in \mathbb{N}\}$. There is a sequence $\{x_{nm}(k_m)\}_{m \in \mathbb{N}}$ converging to $y$, where each $k_m < k_{m+1}$. This is a contradiction because $\{x_{nm}(k_m): m \in \mathbb{N}\}$ is closed.

Hence, $X$ is a closed, countable-to-one image of a metric space. $\square$

Example 9. There is a closed image of a countable metric space, which is not determined by a countable cover of metric subsets.
**Example 15.** Let \( X = \{ (x, y): x, y \in \mathbb{Q} \} \) be endowed with the subspace topology of \( \mathbb{R}^2 \) with the usual topology. Then \( X \) is a countable metric space. Let \( A = \{ (x, 0): x \in \mathbb{Q} \} \). And let \( Y = X/A \) be the quotient space from \( X \) by identifying all the points of \( A \). Then \( Y \) is a closed image of a countable metric space. But \( Y \) is not determined by a countable cover of metric subsets by [12, Example 1.5(1)]. \( \square \)

**Question 10.** How does one characterize, in intrinsic terms, closed, countable-to-one images of metric spaces?

3. Examples

In this section, two questions about open maps are negatively answered.

**Question 11.** [11] Does every open map preserve a weakly first-countable space?

We shall give an example which shows that an open, countable-to-one map may not preserve a weakly first-countable space.

**Lemma 12.** Let \( \mathbb{R} \) be the real numbers with the usual topology. Then \( \mathbb{R} \) has \( \omega_1 \) many disjoint dense subsets.

**Proof.** For each \( r \in \mathbb{R} \), put \( r + \mathbb{Q} = \{ r + q: q \in \mathbb{Q} \} \). Pick \( p_1 \in \mathbb{P} \), then \( p_1 + \mathbb{Q} \) is a dense subset that is disjoint with \( \mathbb{Q} \). For \( \alpha < \omega_1 \), assume we have selected out disjoint dense subsets \( \{ p_\beta + \mathbb{Q}: \beta < \alpha \} \). Let \( A = \mathbb{R} \setminus \bigcup \{ p_\beta + \mathbb{Q}: \beta < \alpha \} \), pick \( p_\alpha \in A \cap \mathbb{P} \), then \( (p_\alpha + \mathbb{Q}) \cap (p_\beta + \mathbb{Q}) = \emptyset \) for each \( \beta < \alpha \). Otherwise, there are \( r_1, r_2 \in \mathbb{Q} \) such that \( p_\alpha + r_1 = p_\beta + r_2 \), so \( p_\alpha = p_\beta + r_2 - r_1 \in p_\beta + \mathbb{Q} \), a contradiction. In this way, we obtain \( \omega_1 \) many disjoint dense subsets \( \{ p_\alpha + \mathbb{Q}: \alpha < \omega_1 \} \).

Let \( S_\kappa \) be the quotient space by identifying all limit points of the topological sum of \( \kappa \) many convergent sequences.

**Example 13.** There is an open map from a countable space with a countable weak base onto \( S_{\omega_1} \).

**Proof.** Let \( R = \bigcup \{ p_1 + \mathbb{Q}: i \in \mathbb{N} \} \), where \( \{ p_1 + \mathbb{Q}: i \in \mathbb{N} \} \) are disjoint dense subsets of \( \mathbb{R} \) by Lemma 9. We write \( p_1 + \mathbb{Q} = \{ p_1 + r_n: n \in \mathbb{N} \} \). For each \( p_1 + r_n \), take a sequence \( \{ x_j(p_1, r_n) \} \) which converges to a point \( x(p_1, r_n) \) in \( \mathbb{R}^2 \). Let \( M \) be the topological sum \( R \oplus \bigoplus \{ x_j(p_1, r_n): j \in \mathbb{N} \} \cup \{ x(p_1, r_n): i, n \in \mathbb{N} \} \). And let \( X \) be the quotient space of \( M \) by identifying \( x_j(p_1, r_n) \) and \( p_1 + r_n \) to a point. Then \( X \) is a quotient, two-to-one image of the countable metric space \( M \), hence \( X \) is a countable space with a countable weak base [8]. We write \( S_{\omega_1} = \{ \infty \} \cup \{ z_j(i): i, j \in \mathbb{N} \} \), where \( z_j(i) \to \infty \) for each \( i \in \mathbb{N} \). Define \( f: X \to S_{\omega_1} \) as follows: \( f(R) = \{ \infty \} \), \( f(x_j(p_1, r_n)) = z_j(i) \) for each \( n \in \mathbb{N} \). It is not difficult to see that \( f \) is an open map.

Since \( S_{\omega_1} \) is not weakly first-countable [8], it does not hold that spaces with weakly first-countability are preserved by open maps. \( \square \)

Gruenhage et al. [4] proved that quotient \( s \)-images of metric spaces are preserved by quotient, countable-to-one maps; and pseudo-open, \( s \)-images of metric spaces are preserved by open, \( s \)-maps. They asked the following question in [4].

**Question 14.** Are quotient \( s \)-images of metric spaces preserved by open, \( s \)-maps?

We shall give a negative answer to this question by the following example, which also shows that an open compact map may not preserve a weakly first-countable space. This is another negative answer to Question 11.

**Example 15.** There is an open compact map from a quotient, two-to-one image of a metric space onto \( S_{\omega_1} \).

**Proof.** Let \( \{ p_\alpha + \mathbb{Q}: \alpha < \omega_1 \} \) be disjoint families of dense subsets of \( \mathbb{R} \). Let \( \{ x \in [0, 1]: x \in p_\alpha + \mathbb{Q} \} = \{ p_\alpha + r_n: n \in \mathbb{N} \} \). For each \( \alpha < \omega_1 \) and \( n, j \in \mathbb{N} \), let \( x_j(p_\alpha, r_n) = (p_\alpha + r_n, 1/j) \) and \( x(p_\alpha, r_n) = (p_\alpha + r_n, 0) \). Then \( x_j(p_\alpha, r_n) \to x(p_\alpha, r_n) \) in \( \mathbb{R}^2 \). For \( \alpha < \omega_1 \), let \( M_\alpha = \bigcup_{n \in \mathbb{N}} \{ x_j(p_\alpha, r_n): j \in \mathbb{N} \} \cup \{ x(p_\alpha, r_n) \} \cup \{ x_\alpha(j): \alpha < \omega_1 \} \).
\( \omega_1, j \in \mathbb{N} \), here each \( x_\alpha(j) \in \mathbb{R}^2 \). Define a topology on \( M_\alpha \) as follows: each \( x_j(p_\alpha, r_n) \) is an isolated point; an element of a local base of \( x_\alpha(j) \) in \( M_\alpha \) has the form \( \{ x_\alpha(j) \} \cup \{ x_j(p_\alpha, r_n): n \geq m \}, \forall m \in \mathbb{N} \); an element of a local base of \( x(p_\alpha, r_n) \) in \( M_\alpha \) has the form \( \{ x(p_\alpha, r_n) \} \cup \{ x_j(p_\alpha, r_n): j \geq m \}, \forall m \in \mathbb{N} \). It is easy to see that \( M_\alpha \) is a countable, regular and first-countable space, hence it is a metrizable space. Let \( M \) be the topological sum of \( \{ M_\alpha: \alpha < \omega_1 \} \).

Let \( X \) be the quotient space of a topological sum \([0, 1] \oplus M\) by identifying \( x(p_\alpha, r_n) \) and \( p_\alpha + r_n \) to a point. Then \( X \) is a quotient, two-to-one image of a metric space. Thus \( X \) is also a weakly first-countable space [8].

We write \( S_{\omega_1} = \{ \infty \} \cup \{ x_j(\alpha): j \in \mathbb{N}, \alpha < \omega_1 \} \), where \( x_j(\alpha) \to \infty \) for each \( \alpha < \omega_1 \). Define \( f: X \to S_{\omega_1} \) by \( f([0, 1]) = \{ \infty \}, f(\{ x_j(p_\alpha, r_n): n \in \mathbb{N} \} \cup \{ x_\alpha(j) \}) = \{ x_j(\alpha) \} \). It is easy to see that \( f \) is an open, compact, \( s \)-map.

Since \( S_{\omega_1} \) is not any quotient, \( s \)-image of a metric space [6], it shows that an open, \( s \)-map may not preserve a quotient, \( s \)-image of a metric space. It is also proved that an open, compact map may not preserve a weakly first-countable space. \( \square \)

References