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Closed Images of some Kinds of Generalized Countably Compact Spaces

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This paper discusses closed images of some generalized countably compact spaces and obtaines the following results:

1) The image of a β-space under a closed mapping is a β-space.

2) The image of a $W\sigma$-space under a closed mapping is a $W\sigma$-space.

3) The image, which is a q-space, of a WN-space under a closed mapping is a WN-space.

In 1962, R.W.Heath [1] introduced the map g. Let (X, τ) be a topological space, and N the set of natural numbers. Map g satisfies that for each $x \in X$, $x \in \bigcap_{n \in N} g(n, x)$; $g(n+1, x) \subseteq g(n, x)$. Later, in [2], Heath and Hodel used this kind of map to characterize a large number of generalized metric spaces. These maps are called Heath-Hodel maps.

By using Heath-Hodel maps, Gittings proved that the majority of general-

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ized metric spaces can be preserved by finite-to- one open maps. As for the case of closed maps, though some topologists have discussed it, the results obtained are unsatisfactory. Sometimes it may not be convenient to use the Heath-Hodel maps to discuss the closed images of the spaces charactrerized by them. Thus, before proving each mapping theorem in this paper, we will characterize the corresponding space in terms of a certain sequence of closed sets.

All spaces in this paper are required to be T_1, maps are continuous subjective, and $\{x_n\}$ denotes the sequence whose n-th term is x_n.

DEFINITION 1 [5]. Let (X,τ) be a topological space and $g : N \times X - \tau$ a Heath-Hodel map satisfying the property that if for every $n \in N$, $p \in g(n.x_n)$, where $p \in X$, then $\{x_n\}$ has cluster points. Then X is called a β-space and g a β-function of X.

Lemma 1. For space (X, τ), the following are equivalent:

a) X is a β-space;

b) For every open set U of X, there is a closed set sequence $\{F_n(U)\}$ corresponding to it such that

1) $F_n(U) \subseteq F_{n+1}(U), F_n(U) \subseteq U$,

2) If V is another open set of X and $U \subseteq V$, then $F_n(U) \subseteq F_n(V)$,

3) If $\{U_n\}$ is an increasing open cover of X, then $\bigcup_{n\in N} F_n(U_n) = X$;

c) For each closed set F of X, there is an open set sequence $\{U_n(F)\}$ such that

1) $U_n(F) \supseteq U_{n+1}(F), U_n(F) \subseteq F$,

2) If G is nother closed set of X and $F \subseteq G$, then $U_n(F) \subseteq U_n(G)$,

3) If $\{F_n\}$ is a decreasing closed set sequence with empty intersection, then $\bigcap_{n\in N} U_n(F_n) = \phi$.

PROOF. $b) \Leftrightarrow c)$ is obvious. We will prove only $a) \Leftrightarrow c)$.

$a) \Leftrightarrow c)$. Let g be a β-function. For any closed set F of X, let

$$U_n(F) = U\{g(n,x) : x \in F\},$$

$\{U_n(F)\}$ corresponds to F and satisfies 1)-3) of c). In fact, 1) and 2) are obvious. If there exists a decreasing closed set sequence $\{F(n)\}$ of X and $\bigcap_{n\in N} F_n = \phi$ such that $\bigcap_{n\in N} U_n(F_n) \neq \phi$, i.e., there is a point $p \in \bigcap_{n\in N} U_n(F_n)$, then for each $n \in N$, there is an $x_n \in F_n$ such that $p \in g(n, x_n)$. Because g is a β-function, $\{x_n\}$ has cluster points. However, $\{F_n\}$ is decreasing and F_n is closed, so we have $\bigcap_{n\in N} F_n \neq \phi$, a contradiction.

$c) \Leftrightarrow a)$. Assume that for every closed set F of X and $n \in N$, there is $\{U_n(F)\}$ as in c). For $x \in X$ and $n \in N$, let $g(n,x) = U_n(\{x_n\})$. It is easy to see that g degined in this way is a Heath-Hodel map. We now check that g is a β-function of X. Let $p \in X$and for each $n \in N, p \in g(n, x_n)$. If $\{x_n\}$ has no cluster points, let $F_n = \{x_i : i \geq n\}$; then $\{F_n\}$ is a decreasing closed set sequence with $\bigcap_{n \in N} F_n = \phi$. Hence, by 3 of c), $\bigcap_{n \in N} U_n(F_n) = \phi$. But by 2) of c), $U\{g(n,x) : x \in F_n\} \sqsubseteq U_n(F_n)$ and $p \in \bigcap_{n \in N}(U\{g(n,x) : x \in F_n\})$, a contradiction.

THEOREM 1. The image of a β-space under a closed mapping is a β-space.

PROOF. Assume that $f : X \to Y$ is a closed mapping and X a β-space. For any open set U of X, $\{F_n(U)\}$ corresponding to U satisfies b) of Lemma 1. For every open set V of Y, let $G_n(V) = f(F_n(f^{-1}(V)))$. Since f is a closed mapping, $G_n(V)$ is closed in Y. Let $\{G_n(V)\}$ correspond to V. It is not difficult to chekc that this correspondence satisfies b) of Lemma 1. Hence, Y is a β-space.

DEFINITION 2 [6]. Let (X, τ) be a topological space, and $g : N \times X$ a Heath-Hodel map of X. If g satisfies that if for each $n \in N$ and $p \in X$, $p \in g(n, y_n)$, $y_n \in g(n, x_n)$, then $\{x_n\}$ has cluster points, then X is called a $W\sigma$-space and g is a $W\sigma$- function of X.

LEMMA 2. For a space (X, τ), the following are equivalent:

a) X is a $W\sigma$-space;

b) For an arbitrary set S of X, there is an open set sequence $\{U(S)\}$ of X such that

1)$U_1 \sqsubseteq U_n(S), S \sqsubseteq U_n(S)$,

2) if $S \sqsubseteq T \sqsubseteq X$, then $U_n(S) \sqsubseteq U_n(T)$,

3) If $\{F\}$ is a decreasing closed set sequence with empth intersection, then $\bigcap_{n \in N} U_n(U_n(F_n)) = \phi$;

c) For an arbitrary set S of X, there is a closed set sequence $\{F(S)\}$ of X such that

1)$F_n(S) \sqsubseteq F_{n+1}(S), F_n(S) \sqsubseteq S$,

2) if $S \sqsubseteq T \sqsubseteq X$, then $F_n(S) \sqsubseteq F_n(T)$,

3) if $\{U_n\}$ is an increasing open set sequence with then $\bigcup_{n \in N} F_n(F_n(U_n)) = X$.

PROOF. $b) \Rightarrow c)$ is obvious.

$a) \Rightarrow b)$. Let g be a $W\sigma$-function of X. For each set S of X, let $U_n(S) = U\{g(n,x) : x \in S\}$ and $\{U_n(S)\}$ correspond to S. It is easy to see that $\{U_n(S)\}$ fits 1), 2) of b). If $\{F_n\}$ is a decreasing closed set sequence of X such that $\bigcup_{n \in N} F_n = \phi$ but $\bigcap_{n \in N} U_n(U_n(F_n)) \neq \phi$, i.e., there exists a point p of X such that $p \in \bigcap_{n \in N} U_n(U_n(F_n))$, then for each $n \in N$, there are $x_n \in F_n$, $y_n \in (n, x_n)$

and $p \in (n, y_n)$. It folows that $\{x_n\}$ has cluster points and this contradicts $\bigcap_{n\in N} F_n \neq \phi$.

$b) \Rightarrow a)$. Assume that for each set S of X, there is $\{U_n(S)\}$ as inb). For $n \in N$ and $x \in X$, let $g(n,x) = U_n(\{x\})$. At first, g is a Heath-Hodel map of X. To check that g is a $W\sigma$-function, we prove only that if $p \in X$, $p \in g(n, y_n)$, $y_n \in g(n, x_n)$,$n \in N$, then $\{x_n\}$ has cluster points. If not, let $F_n = \{x_i : i \geq n\}$; then $\{F_n\}$ is a decreasing closed set sequence and $\bigcap_{n\in N} F_n = \phi$, so $\bigcap_{n\in N} U_n(U_n(F_n)) = \phi$. However, by 2) of b),

$$\bigcup\{g(n, x) : x \in F_n\} \subseteq U_n(F_n),$$
$$\bigcup\{g(n, y) : y \in \bigcup\{g(n, x) : x \in F_n\}\} \subseteq U_n(\bigcup\{g(n, x) : x \in F_n\}) \subseteq U_n(U_n(F_n))$$

and it is easy to see $p \in \bigcap_{n\in N}(U\{g(n,y) : y \in U\{g(n,x) : x \in F_n\}\})$.a contradiction.

THEOREM 2. The image of a $W\sigma$-space under a closed mapping is a $W\sigma$-space.

PROOF. Let X be a $W\sigma$-space, and $f : X \to Y$ a closed mapping. For any set S of X, a closed set sequence $\{F_n(S)\}$ is as in c) in Lemma 2. For every set T of Y, let $G_n(T) = f(F_n(f^{-1}(T)))$. Then $G_n(T)$ is closed in Y and $G_n(T) \subseteq G_{n+1}(T)$; if $T_1 \subseteq T_2 \subseteq Y$, then $G_n(T_1) \subseteq G_n(T_2)$. If $\{V_n\}$ is an increasing open cover of Y, then $\{f^{-1}(V)\}$ is an open cover of X, so $\bigcup_{n\in N} F_n(F_n(f^{-1}(V_n)) = X$. Since $G_n(V_n) = f(F_n(f^{-1}(V_n)))$, we have $f^{-1}(G_n(V_n)) \sqsupseteq F_n(f^{-1}(V_n))$;thus $G_n(G_n(V_n)) = f(F_n(f^{-1}(G_n(V_n))) \sqsupseteq f(F_n(F_n(f^{-1}(V_n))))$, and then $\bigcup_{n\in N} G_n(G_n(V_n)) = Y$. By c) in Lemma 2, Y is a $W\sigma$- space.

DEFINITION 3 [7]. Let (X, τ) be a topological space, $g : N \times x \to \tau$ a Heath-Hodel map. If g satisfies the property that if for each $n \in N$, $g(n,p) \cap g(n, x_n) \neq \phi$. where $p \in X$, then $\{x_n\}$ has cluster points, then X is called a WN-space and g a WN- function.

DEFINITION 4 [8]. If $g : N \times X \to \tau$ is a Heath-Hodel map of X satisfying the property that if $x_n \in g(n,p)$, $p \in X$, then $\{x_n\}$ has cluster points, then X is called a q-space and g a q-function of X.

LEMMA 3. For a q-space (X, τ), the following are equivalent:

a) X is a WN-space;

b) For every open set U of X, there is a closed set sequence $\{F_n(U)\}$ corresponding to it such that

1)$F_n(U) \sqsubseteq F_{n+1}(U)$,$F_n(U) \sqsubseteq U$,

2) if U, V are open sets of X and $U \subseteq V$, then $F_n(U) \subseteq F_n(V)$,

3) if $\{U_n\}$ is an increasing open cover of X, then $\bigcup_{n\in N} Int F_n(U_n) = X$;

c) For every closed set F of X, there is an open set sequence $\{U_n(F)\}$ corresponding to it such that

1)$U_{n+1}(F) \sqsubseteq U_n(F), F \sqsubseteq U_n(F)$,

2) if F, G are closed sets of X and $F \subseteq G$, then $U_n(F) \subseteq U_n(G)$,

3) if $\{F_n\}$ is a decreasing closed set sequence with $\bigcap_{n\in N} F_n = \phi$, then $\bigcap_{n\in N} U_n(\bar{F_n}) = \phi$.

PROOF. We prove only $a) \Leftrightarrow c)$.

$a) \Rightarrow c)$. Assume g is a WN-function. For any closed set F and $n \in N$, let $U_n(F) = U\{g(n,x) : x \in F\}$. In the following, we check only that the defined $U_n(F)$ meets 3) in c). If there exists a decreasing sequence $\{F_n\}$ of closed sets and $\bigcap_{n\in N} F_n = \phi$, but $\bigcap_{n\in N} U_n(\bar{F_n}) \neq \phi$, i.e., for some $p \in X$, $p \in \bigcap_{n\in N} U_n(\bar{F_n})$, then for every $n \in N$, there is $x_n \in F_n$ such that $g(n,p) \cap g(n,x_n) \neq \phi$ and $\{x_n\}$ must have cluster point. This contradicts $\bigcap_{n\in N} F_n = \phi$.

$c) \Rightarrow a)$. Assume that for a closed set F of X, $\{U_n(F)\}$ corresponds to F as in c). Because X is a q- space, let h be a q-function of X. For $x \in X$ and $n \in N$, let $g(n,x) = U_n(\{x\}) \cap h(n,x)$. It is easy to see that g is a Heath-Hodel map. If $p \in X$, $g(n,p) \cap g(n,x_n) \neq \phi$, but $\{x_n\}$ has no cluster points. Let $F_n = \{x_i : i \geq n\}$, then $\{F_n\}$ is a decreasing sequence of closed sets such that $\bigcap_{b\in N} F_n = \phi$, so $\bigcap_{n\in N} U_n(\bar{F_n}) \neq \phi$. However, $U\{g(n,x) : x \in F_n\} \subseteq U_n(F)$ and for each $n \in N$, there exists $y_n \in g(n,p) \cap g(n,x_n)$; thus $\{y_n\}$ has cluster points and $\bigcap_{n\in N} U_n(\bar{F_n}) \neq \phi$, a contradiction.

REMARK. From the proof of $a) \Rightarrow c)$, we can see that every WN-space is countably paracompact.

LEMMA 4. Let X be a countably paracompact space, Y a q-space and $f : X \to Y$ a closed mapping. Then for every point $y \in Y$, $Brf^{-1}(y)$ is countably compact.

The proof of Lemma 4 needs the following fact:

If X is countably paracompact, $\{x_n : nN\}$ is discrete and closed, then there exists a locally finite collection $\{U(x_n) : n \in N\}$ of open sets such that $x_n \in U(x_n)$. Now, this lemma can be proved as the corresponding result in [8].

THEOREM 3. The image, which is a q-space, of a WN-space is a WN-space.

PROOF. For every point $y \in Y$, pick a $p_y \in f^{-1}(y)$. Let

$$C_y = \begin{cases} B_r f^{-1}(z), & if \ \ B_r f^{-1} \neq \phi, \\ p_y, & if \ \ B_r f^{-1} = \phi, \end{cases}$$

Let $C = U\{C_y : y \in Y\}$, $g = f|C$. Then it is easy to see that C is a closed set of X and $g(C) = Y$. Thus we can assume, without loss of generality, that f is quasi-perfect mapping. By b) of Lemma 3, to prove Y is a WN-space, it is sufficient to find $G_n(V)$ for each open set V and $n \in N$ which satisfies every condition in b). For this, it is enough to let $G_n(V) = f(F_n(f^{-1}(V)))$, where $F_n(\cdot)$ is the closed set of X as in b). The work to check that $G_n(V)$ meets b) is left to the reader. Thus Y is a WN-space.

REMARK. Lutzer gave an example (Example 4.3 in [9]) which shows that the perfect image of a first countable stratifiable space need not be a q-space. so the requirement that "Y is a q-space" cannot be ommitted.

Before finishing this paper, we will give an application of Lemmas 1 and 3. Definitions of P-space and P*-space can be found in [10] and [11].

THEOREM 4. A β-space is a P-space.

THEOREM 5. A WN-space is a P*-space.

PROOF. We prove only Theorem 4; Theorem 5 can be proved in the same way.

Let X be a β-space. By Lemma 1, for an open set U of X, there is a closed set sequence $\{F_n(U)\}$ corresponding to it as in b) in Lemma 1. Assume that $\{G(\alpha_1, \alpha_2, \cdots, \alpha_n) : \alpha_1, \alpha_2, \cdots, \alpha_n \in \Omega, n \in N\}$ is an increasing collection of open sets of X. For each n N and $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \Omega^n$, let $F(\alpha_1, \alpha_2, \cdots, \alpha_n) = F_n(G(\alpha_1, \alpha_2, \cdots, \alpha_n))$. Then by 1) of b),$F(\alpha_1, \alpha_2, \cdots, \alpha_n) \sqsubseteq G(\alpha_1, \alpha_2, \cdots, \alpha_n)$. By 3) of b), if $(\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots) \in \Omega^\omega$, $\bigcup_{n \in N} G(\alpha_1, \alpha_2, \cdots, \alpha_n) = X$, then

$$\bigcup_{n \in N} F_n(G(\alpha_1, \alpha_2, \cdots, \alpha_n)) = X,$$

i.e.,

$$\bigcup_{n \in N} F(\alpha_1, \alpha_2, \cdots, \alpha_n) = X.$$

Thus X is a P-space.

Theorems 4,5 improve the corresponding results in [7].

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