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NOTES ON SEQUENTIALLY CONNECTED SPACES

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Abstract. We discuss the relationship between two different sequential connectedness, and prove that sequential connectedness is countably multiplicative.

1. Introduction

In recent years, many topologists concentrate their attention to connectedness. V. V. Tkachuk [7] raised the question: Can a connected sequential space be characterized as a quotient image of a connected metric space? A. Fedeli and A. Le Donne [4] answered this. They introduced sequentially connected spaces by the concept of sequentially open sets, and precisely described the continuous images of connected metric spaces as sequentially connected spaces [6]. This implies the importance of sequential connectedness. Á. Császár [2] defined γ-open sets with the help of a class of universal set-valued functions, and introduced the concept of γ-connected sets. Sequentially open sets are a class of special γ-open sets, therefore the theory of γ-connected sets is also adapt to sequentially connected spaces. It is well known that connectedness can be characterized by a pair of disjoint open sets, disjoint closed sets or separated sets, respectively. We notice that γ-connected subsets are given by γ-separated subsets, so it is natural to ask whether the connectedness defined by γ-separated sets is consistent with the one defined by a pair of disjoint γ-open sets. On the other hand, as we know, connectedness is arbitrary multiplicative (i.e., any Cartesian product space of connected spaces is a connected space). However in [2] the author did not discuss the product property of γ-connectedness, so it is interesting to study whether γ-connectedness is a multiplicative property.

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In this short note, we discuss the above two problems on sequential connectedness, analyze some relationship between the two sequential connectedness defined by sequentially separated sets or a pair of disjoint sequentially open sets, give examples to explain the difference between them, and further prove that sequential connectedness is countably multiplicative.

We refer the reader to [3] for notations and terminology not explicitly given here.

2. Preliminaries

First recall the concept of γ-connected sets and some related definitions. Let X be a set and let $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ be a monotonic set-valued function, that is, $A \subset B \subset X$ implies $\gamma(A) \subset \gamma(B)$. A subset A in X is *γ-open* if $A \subset \gamma(A)$. Clearly, any union of γ-open sets is γ-open. The collection of all γ-open sets in a set X is a generalized topology in the sense of [1]. The complement of a γ-open set is said to be a *γ-closed* set. According to [2], for $A \subset X$, $\bigcap\{F : A \subset F$ and F is a γ-closed set in $X\}$ is called a *γ-closure* of A in X, and is denoted by $c_\gamma A$. It is clear that $c_\gamma A$ is a γ-closed set of X. Given $U, V \subset X$, U and V are *γ-separated* in X if $c_\gamma U \cap V = c_\gamma V \cap U = \emptyset$. A subset $S \subset X$ is called *γ-connected* if S cannot be expressed as the union of two nonempty γ-separated sets of X [2].

A. Fedeli and A. Le Donne [4] defined sequential connectedness in a different manner. Let X be a topological space. For $P \subset X$, P is a *sequential neighborhood* of x in X if every sequence converging to x is eventually in P. P is *sequentially open* in X if P is a sequential neighborhood of x in X for each $x \in P$. P is *sequentially closed* in X if $X \setminus P$ is sequentially open. A subspace S of X is said to be *sequentially connected* if S cannot be expressed as the union of two nonempty disjoint sequentially open sets of S [4].

It is worth noting that γ-connected subsets are defined in any set, and sequentially connected subsets are defined in topological spaces. A sequentially open set in a topological space is a special γ-open set. In fact, let $\gamma(A) = \{x \in A : A$ is a sequential neighborhood of x in $X\}$ for every $A \subset X$. Then the set-valued function $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ is monotonic, and a subset $A \subset X$ is sequentially open iff $A \subset \gamma(A)$. This shows that A is sequentially open in X iff A is γ-open. In this paper, the γ-closures, γ-separated sets, and γ-connected sets generated by the above function γ defined by sequential neighborhoods will be called s-closures, s-separated sets and s-connected sets, respectively, and denote the s-closure of a subset A in X by $c_s(A)$. For a subspace Y of X and $A \subset Y$, let $c_{s,Y}(A) = \bigcap\{F : A \subset F$ and F is sequentially closed in $Y\}$. Then we say that $c_{s,Y}(A)$ is the s-closure of A in Y. Clearly, $c_{s,Y}(A) \subset c_s(A) \cap Y$.

Obviously, X is a sequentially connected space if and only if it is s-connected. The results on γ-connectedness obtained by Á. Császár [2] hold for s-connectedness, and also hold for sequentially connected spaces. For example, if S is an s-connected subset of X and $S \subset Y \subset c_s(S)$, then the subset Y is s-connected in X by Theorem 1.4 in [2].

3. Some examples

LEMMA 3.1. *Let X be a topological space. If Y is sequentially closed in X and A is sequentially closed in Y, then A is sequentially closed in X.*

PROOF. According to [5], A is sequentially closed in X if and only if $x \in A$ whenever a sequence $\{x_n\}$ in A converges to x in X. Now, let $\{x_n\} \subset A$ be a sequence converging to x in X. Since Y is sequentially closed in X and $A \subset Y$, then $x \in Y$. Thus $\{x_n\}$ converges to x in Y. Moreover, A is sequentially closed in Y, therefore $x \in A$. So A is sequentially closed in X. □

THEOREM 3.2. *The following properties hold for a topological space X.*

(1) *If S is a sequentially connected subset of X, then S is s-connected in X.*

(2) *If S is s-connected and sequentially closed in X, then S is sequentially connected in X.*

PROOF. (1) Suppose that S is not s-connected in X. There exist nonempty s-separated subsets A, B of X such that $S = A \bigcup B$. Since $c_{s,S}(A) \subset c_s(A) \cap S$, and $c_{s,S}(B) \subset c_s(B) \cap S$, it is easy to prove that A and B are s-separated subsets of S as well. Therefore, A and B are disjoint sequentially closed subsets of S, so S is not sequentially connected, which is a contradiction.

(2) Assume that S is not sequentially connected in X. Then there exist nonempty disjoint sequentially closed subsets A and B of S such that $S = A \bigcup B$. By Lemma 3.1, A and B are sequentially closed subsets of X. So A and B are s-separated subsets in X. Therefore, S is not s-connected. This is a contradiction. □

For a subset S of a topological space X, we have the following properties by Theorem 3.2: (1) $c_s(S)$ is sequentially connected in X if and only if it is s-connected in X; (2) if S is a sequentially connected subset of X, then $c_s(S)$ is sequentially connected. The following example shows that the condition of sequentially closed sets in Theorem 3.2(2) is very important.

EXAMPLE 3.3. There exists a topological space X with an s-connected set which is not sequentially connected.

PROOF. Let $\mathbf{N}$ be the set of positive integers, and let $X = \{0,1\} \cup \big(\mathbf{N} \times [0,1)\big)$ be endowed with the topology as follows.

(1) $\mathbf{N} \times [0,1)$ has the usual topology as an open subspace of X;

(2) Each element of a neighborhood base of 1 in X has the form $\{1\} \cup \big(\mathbf{N} \times (1-1/n, 1)\big)$ for each $n \in \mathbf{N}$;

(3) Each element of a neighborhood base of 0 in X has the form $\{0\} \cup \Big(\bigcup_{x \in W} \big(\{x\} \times [0,1) \setminus F_x\big)\Big)$. Here $W \subset \mathbf{N}$, $\mathbf{N} \setminus W$ is finite and $F_x \subset \{x\} \times (0,1)$ is finite.

Let $T = X \setminus \big(\mathbf{N} \times \{0\}\big)$, $S = T \setminus \{0\}$. For each $n \in \mathbf{N}$, since $(n,0)$ is a limit point of a convergent sequence in S, it follows that $(n,0) \in c_s(S)$. Since the sequence $\big\{(n,0)\big\}$ converges to 0 in X, then $c_s(S) = X$. Because S is the hedgehog space $J(\omega)$ with spininess ω [3], it is a connected metric subspace of X. S is a sequentially connected subset of X as a sequentially open set is equivalent to an open set in a metric space. By Theorem 3.2, S is an s-connected subset of X as well. Moreover, if $S \subset T \subset c_s(S)$, then T is an s-connected subset of X. However, T is not sequentially connected in X. In fact, if a sequence $\{s_i\} \subset S$ converges to 0 in X, then each $\{x\} \times (0,1)$ includes only finitely many terms of $\{s_i\}$, hence there exists a neighborhood of 0 that does not intersect with $\{s_i\}$, a contradiction. It indicates that each sequence converging to 0 in X is only eventually in $\{0\} \cup \big(\mathbf{N} \times \{0\}\big)$. Thus $\{0\}$ is sequentially closed and sequentially open in T. So T is not a sequentially connected subspace of X. □

It is a regrettable fact that the T in Example 3.3 is an s-connected subset in X in the sense of Császár [2], but T is not an s-connected subspace in X in the sense of Fedeli and Le Donne [4]. On the other hand, Example 3.3 also indicates that if S is a sequentially connected subset of X and $S \subset T \subset c_s(S)$, then T cannot be a sequentially connected subset of X. Then, it is natural to ask under what conditions T is sequentially connected?

For $S \subset X$, we denote $cs(S) = \{x \in X\colon$ there is a sequence in S converging to x in $X\}$. Then $S \subset cs(S) \subset c_s(S) \subset \overline{S}$. In Example 3.3, $S = c_{s,T}(S) \neq c_s(S) \cap T$, $S = cs(S) \neq c_s(S)$.

THEOREM 3.4. *Let S be a sequentially connected subset of a space X. If $S \subset T \subset cs(S)$, the following properties hold.*

(1) *T is sequentially connected.*

(2) *If $t \in c_s(S) \setminus cs(S)$ and $\{t\}$ is sequentially closed in X, then $S \cup \{t\}$ is not sequentially connected.*

PROOF. (1) Suppose that there exist disjoint sequentially closed subsets A, B of T such that $T = A \cup B$. Therefore, $A \cap S$ and $B \cap S$ are disjoint sequentially closed subsets of S. Since S is sequentially connected, we may assume that $A \cap S = \emptyset$, then $S \subset B$. If $x \in T$, there exists a sequence $\{x_n\} \subset S$ converging to x in X, then $\{x_n\}$ converges to x in T, $x \in B$ because B is sequentially closed in T, thus $T \subset B$. This shows that $A = \emptyset$, therefore T is sequentially connected.

(2) If $t \in c_s(S) \setminus cs(S)$, then $\{t\}$ is sequentially open in $S \cup \{t\}$. Therefore, if $\{t\}$ is sequentially closed in X, then $\{t\}$ is a nonempty sequentially open and sequentially closed proper subset of $S \cup \{t\}$. So $S \cup \{t\}$ is not sequentially connected. □

Evidently, every s-connected subset of X is connected. The subspace T in Example 3.4 is a connected space, but it is not s-connected. In the following, we construct an example satisfying that condition, which has better separation property.

Example 3.5. There is a connected space X which is not s-connected.

Proof. For the Stone–Čech compactification $\beta\mathbf{R}$ of the real line $\mathbf{R}$, we first prove that there is no sequence in $\mathbf{R}$ converging to p for each $p \in \beta\mathbf{R} \setminus \mathbf{R}$. In fact, suppose that there exists a sequence $\{x_n\} \subset \mathbf{R}$ converging to p. Let $A = \{x_{2n} : n \in \mathbf{N}\}$, $B = \{x_{2n-1} : n \in \mathbf{N}\}$. Then A, B are disjoint and closed in $\mathbf{R}$. Since $\mathbf{R}$ is normal, so $\mathrm{cl}_{\beta\mathbf{R}}(A) \cap \mathrm{cl}_{\beta\mathbf{R}}(B) = \emptyset$ by Corollary 3.6.4 in [3]. However, $p \in \mathrm{cl}_{\beta\mathbf{R}}(A) \cap \mathrm{cl}_{\beta\mathbf{R}}(B)$ as p is the limit point of the sequence $\{x_n\}$, a contradiction.

Now, take $p \in \beta\mathbf{R} \setminus \mathbf{R}$, and let $X = \mathbf{R} \cup \{p\}$ with a subspace topology of $\beta\mathbf{R}$. Since $\mathbf{R}$ is dense and connected in X, it follows that X is a connected space. However, $\{p\}$ is sequentially open and sequentially closed by the proof in paragraph above, so X is not s-connected. □

Since $\mathbf{R}$ is a sequentially connected subset of X, Example 3.5 indicates that the closure of an s-connected subset cannot be s-connected.

4. The product of sequentially connected spaces

In this section, we discuss a product property of sequential connectedness.

Lemma 4.1 [2]. *Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be a cover of a space X, where each S_λ is sequentially connected in X. If $\bigcap_{\lambda \in \Lambda} S_\lambda \neq \emptyset$, then X is sequentially connected.*

Theorem 4.2. *The countable product space of sequentially connected spaces is sequentially connected.*

Proof. First, the sequential connectedness is finitely multiplicative. Let X and Y be sequentially connected spaces. Take a fixed point $p_0 = (x_0, y_0) \in X \times Y$. Let $p = (x, y)$ be any point of $X \times Y$. It is easy to show that sets

$\big(X \times \{y_0\}\big)$, $\big(\{x\} \times Y\big)$ are sequentially connected in $X \times Y$. Moreover, $(x, y_0) \in \big(X \times \{y_0\}\big) \cap \big(\{x\} \times Y\big)$. Thus $\big(X \times \{y_0\}\big) \cup \big(\{x\} \times Y\big)$ is sequentially connected by Lemma 4.1, which contains points p_0, p. From Lemma 4.1, $X \times Y$ is sequentially connected. By induction, a finite product space of sequentially connected spaces is sequentially connected.

Let $\{X_i\}_{i \in \mathbf{N}}$ be a countable family of sequentially connected spaces and let $X = \prod_{i \in \mathbf{N}} X_i$ be the countable product space. Fix a point $\alpha = (\alpha_i) \in X$. For each $n \in \mathbf{N}$, put $P_n = (\prod_{i \leqq n} X_i) \times \big(\prod_{i > n} \{\alpha_i\}\big)$, then $P_n \subset P_{n+1}$. So P_n is a sequentially connected subspace of X by the finite case of the theorem already proved. Let $P = \bigcup_{n \in \mathbf{N}} P_n$, then P is sequentially connected as well by Lemma 4.1, so $c_s(P)$ is a sequentially connected subset of X by Theorem 3.2. In order to prove that X is a sequentially connected space, it will suffice to show that $c_s(P) = X$. For every $\beta = (\beta_i) \in X$ and $n \in \mathbf{N}$, let $x_n \in X$ be the point defined by $x_{n,i} = \beta_i$ for $i \leqq n$, and $x_{n,i} = \alpha_i$ for $i > n$, where $x_{n,i}$ is the i-th coordinate of x_n. Since a convergent sequence in a product space is convergent by coordinates, then $\{x_n\}$ in X converges to β, therefore $c_s(P) = X$. Hence X is sequentially connected. □

QUESTION 4.3. Is sequential connectedness an arbitrary multiplicative property?

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