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A NOTE ON CLOSED IMAGES OF LOCALLY COMPACT METRIC SPACES

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Abstract. A decomposition theorem about closed images of locally compact metric spaces is discussed. It is shown that a space is a closed image of a locally compact metric space if and only if it is a regular Fréchet space with a point-countable k-network, and each of its closed first-countable subset is locally compact.

1. Introduction

To find the internal characterizations of certain images of metric spaces is one of the central questions in general topology. Since A. Arhangel'skiĭ [1], topologists pay close attention to the closed images of various metric spaces, in particular are interested in the question after L. Foged [4] obtained the nice characterization of closed images of metric spaces. The closed images of locally compact metric spaces have been studied extensively in recent years. The following are some related results, which characterize the closed mapping theorems or gain the decomposition theorems of locally compact metric spaces.

LEMMA 1.1. *The following are equivalent for a regular space X:*

(1) *X is a closed image of a locally compact metric space;*

(2) *X is a Fréchet space with a σ-hereditarily closure-preserving compact k-network* [6]*;*

(3) *X is a closed image of a metric space, and each of its closed first-countable subset is locally compact* [11].

It is necessary to obtain some weak characterization theorems or decomposition theorems on closed images of locally compact metric spaces, which

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are a common generalization of Lemma 1.1. k-networks as a successful generalization of the basis of topological spaces, is a powerful tool for studying generalized metric spaces [8, 9].

LEMMA 1.2 [4]. *Every closed image of a metric space has a σ-hereditarily closure-preserving k-network, and every space with a σ-hereditarily closure-preserving k-network has a point-countable k-network.*

In this paper we discuss some basic properties of spaces with a point-countable k-network. First, we prove that a regular Fréchet space X is a closed image of a locally compact metric space if and only if X has a point-countable k-network, and each of its closed first-countable subset is locally compact. Moreover, some related results concerning the closed or quotient s-images of locally compact metric spaces are also obtained.

In this paper all spaces are regular and T_2, all mappings are continuous and onto. For some terms not defined here see [3].

2. Spaces with a point-countable k-network

Suppose that $\mathcal{P}$ is a family of subsets of a space X. $\mathcal{P}$ is said to be point-countable if for each $x \in X$ the set $\{P \in \mathcal{P} : x \in P\}$ is countable. $\mathcal{P}$ is a k-network for a space X if, whenever $K \subset U$ with K compact and U open in X then $K \subset \cup\mathcal{F} \subset U$ for a finite subset $\mathcal{F}$ of $\mathcal{P}$ [5].

LEMMA 2.1 [5]. *A countably compact k-space with a point-countable k-network is compact and metrizable.*

LEMMA 2.2. *Let $\mathcal{P}$ be a point-countable k-network of a space X. If a point x in X has a countable local base, then for each neighborhood U of x there exists a finite subset $\mathcal{F}$ of $\mathcal{P}$ such that $x \in (\cup\mathcal{F})^\circ \subset \cup\mathcal{F} \subset U$.*

PROOF. Suppose the conclusion is false for a neighborhood U of x in X. Let $\{P \in \mathcal{P} : P \cap C_j \neq \emptyset, P \subset U\} = \{P_{ij} : i \in \mathbf{N}\}$ if C_j is a countable subset of U by the point-countability of $\mathcal{P}$. Put $C_0 = \{x\}$. Then P_{10} is not a neighborhood of x in X, there is a sequence C_1 in $U \backslash P_{10}$ converging to x since x has a countable local base. Thus $C_1 \subset \cup\{P_{ij} : 1 \leqq i \leqq n_1, 0 \leqq j \leqq 1\}$ for some $n_1 \in \mathbf{N}$ because $\mathcal{P}$ is a k-network for X. So $\cup\{P_{ij} : 1 \leqq i \leqq n_1, 0 \leqq j \leqq 1\}$ is also not a neighborhood of x. Repeating the process above, inductively choose a sequence C_m in U converging to x and an increasing sequence $\{n_m\}$ of natural numbers with

$$C_m \subset \cup\{P_{ij} : 1 \leqq i \leqq n_m, 1 \leqq j \leqq m\} \backslash \cup\{P_{ij} : 1 \leqq i \leqq n_{m-1}, 0 \leqq j \leqq m-1\}$$

for each $m \in \mathbf{N}$. This shows that each $P \in \mathcal{P}$ meets only finitely many C_m. Let $S = \{x\} \cup (\cup_{m \in \mathbf{N}} C_m)$. Then x has a countable local base in the subspace

S of X, there is a sequence $\{s_m\}$ in S converging to x with each $s_m \in C_m$. Since $\mathcal{P}$ is a k-network for X, there is $P \in \mathcal{P}$ such that $P \subset U$ and P contains infinitely many s_m. Hence P meets infinitely many C_m, a contradiction. □

A family $\mathcal{P}$ of subsets of a space X is called a k-network by countably compact-closures provided that $\mathcal{P}$ is a k-network for X and the closure of each element of $\mathcal{P}$ is countably compact in X. A mapping $f: X \to Y$ is perfect if f is closed and each $f^{-1}(y)$ is compact in X.

THEOREM 2.3. *Suppose a space X has a point-countable k-network. Then X has a point-countable k-network by countably compact-closures if and only if each closed first-countable subspace of X is locally countably compact.*

PROOF. *Necessity.* Let F be a closed first-countable subspace of X. Let $x \in X$. Let $\mathcal{Q}$ be a point-countable k-network by countably compact-closures for F. By Lemma 2.2 there exists a finite subset $\mathcal{F}$ of $\mathcal{P}$ such that $x \in \operatorname{int}_F(\cup\mathcal{F})$ and $\cup\{\overline{F} : F \in \mathcal{F}\}$ is a countably compact subspace of F. Hence F is locally countably compact.

Sufficiency. Without loss of generality we can assume that $\mathcal{P}$ is a point-countable k-network for X which is closed under finite intersections. Let $\mathcal{H} = \{P \in \mathcal{P} : \overline{P}$ is countably compact in $X\}$. Then $\mathcal{H}$ is a k-network by countably compact-closures for X. In fact, let $K \subset U$ with K compact and U open in X. By Miščenko's Lemma (3.12.23(f) in [3]) there are at most ω irreducible (i.e., not containing proper subcovers) finite covers of K by members of $\mathcal{P}$, say $\{\mathcal{P}_i\}$. Put $\mathcal{A}_n = \wedge_{i \leqq n}\mathcal{P}_i$, and $A_n = \cup\mathcal{A}_n$ for each $n \in \mathbf{N}$. Then $\{A_n\}_{n\in\mathbf{N}}$ is a decreasing net of K in X. If no $\overline{A}_n$ is countably compact in X, each $\overline{A}_n$ contains an infinite closed discrete subset D_n. Define $C = K \cup (\cup_{n\in\mathbf{N}} D_n)$, and endow C with the subspace topology of X. Then no neighborhood of K in C is countably compact. By the compactness of K, C is not locally countably compact. Let $f: C \to C/K$ be a natural quotient mapping. Then f is a perfect mapping and C/K is first-countable (in fact, a metric space), C is a non-locally countably compact, closed first-countable subset of X (in fact, a metric space: apply Lemma 1.2 to K), a contradiction. Thus, $\overline{A}_m$ is countably compact in X for some $m \in \mathbf{N}$. Choose $n \geqq m$ with $K \subset A_n \subset U$. This shows that $\mathcal{A}_n$ is a finite subset of $\mathcal{H}$ and $K \subset \cup\mathcal{A}_n \subset U$. Thus $\mathcal{H}$ is a k-network by countably compact-closures for X. □

Let $\mathcal{P}$ be a family of subsets of a space X. $\mathcal{P}$ is said to be a closed (resp. countably compact, compact or separable) k-network if $\mathcal{P}$ is a k-network for X, and each element of $\mathcal{P}$ is closed (resp. countably compact, compact or separable) in X.

COROLLARY 2.4 [2]. *Let Y have a point-countable closed k-network.*

(1) *If each closed metric subset of Y is locally compact, Y has a point-countable countably compact k-network;*

(2) *If each closed first-countable subset of Y is locally countably compact and each of its countably compact subset is compact, Y has a point-countable compact k-network.*

3. The closed images of metric spaces

In this section we shall give some new characterizations of closed images of locally compact metric spaces by spaces having point-countable k-networks.

LEMMA 3.1 [10]. *A space is a closed image of a locally separable metric space if and only if it is a Fréchet space with a point-countable separable k-network.*

THEOREM 3.2. *A space is a closed image of a locally compact metric space if and only if it is a Fréchet space with a point-countable k-network, and each of its closed first-countable subspace is locally compact.*

PROOF. Necessity is a corollary of Lemma 1.1 and Lemma 1.2. Sufficiency is proved as follows. Suppose that X is a Fréchet space with a point-countable k-network, and each of its closed first-countable subspace is locally compact. By Theorem 2.3, Lemma 2.1 and Lemma 3.1, X is a closed image of a locally separable metric space. By Lemma 1.1 once again, X is a closed image of a locally compact metric space. □

Finally, some corollaries are given on the closed or quotient and s-images of locally compact metric spaces. Let $f : X \to Y$ be a mapping. f is called an s-mapping if each $f^{-1}(y)$ is separable in X. Let $\mathcal{P}$ be a family of subsets of a space X. $\mathcal{P}$ is a cs^*-network for X if, whenever $\{x_n\}$ is a sequence converging to $x \in X$ and U is a neighborhood of x in X, there are $P \in \mathcal{P}$ and a subsequence $\{x_{n_i}\}$ such that $\{x\} \cup \{x_{n_i} : n \in \mathbf{N}\} \subset P \subset U$ [9].

LEMMA 3.3 [7]. *A space X is a quotient s-image of a locally compact metric space if and only if X is a k-space with a point-countable compact k-network.*

COROLLARY 3.4. *The following are equivalent for a Fréchet space X:*

(1) *X is a closed s-image of a locally compact metric space;*

(2) *X is a quotient s-image of a locally compact metric space;*

(3) *X is a quotient s-image of a metric space and each of its closed first-countable subspace is locally compact;*

(4) *X has a point-countable closed k-network and each of its closed first-countable subspace is locally compact;*

(5) *X has a point-countable cs^*-network and each of its closed first-countable subspace is locally compact.*

PROOF. First cite some related results about k-networks, cs^*-networks and closed s-images of metric spaces as follows.

(a) X is a closed s-image of a locally compact metric space if and only if X is a closed s-image of a metric space and each of its closed first-countable subspace is locally compact [11],

(b) X is a closed s-image of a metric space if and only if X is a closed image of a metric space and it has a point-countable cs^*-network [8],

(c) If $\mathcal{P}$ is a point-countable cs^*-network for X, then $\mathcal{P}$ is a k-network for X if and only if each compact subset of X is a sequential subspace [8],

(d) Every quotient s-image of a metric space has a point-countable cs^*-network [8].

Now the corollary is shown as follows. It is obvious that (1) $\Rightarrow$ (2), and (4) $\Rightarrow$ (5). (2) $\Rightarrow$ (4) and (3) by Lemma 3.3, Theorem 2.3, and Lemma 2.1. (3) $\Rightarrow$ (5) by (d). (5) $\Rightarrow$ (1) by (a)–(c) and Theorem 3.2. $\square$

EXAMPLE 3.5. Let S_{ω_1} denote the quotient space which is obtained from the topological sum of ω_1 non-trivial convergent sequences by identifying all limit points. Then S_{ω_1} is a closed image of a locally compact metric space, but it has no point-countable cs^*-network [8].

QUESTION 3.6. Suppose that X is a quotient s-image of a metric space. Is X a quotient s-image of a locally compact metric space if each of its closed first-countable subspace is locally compact?

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