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ON NORMAL SEPARABLE $\aleph$-SPACES

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1. Introduction and definition

It is known to all that separability is equivalent to Lindelöf property in metrizable spaces. The problem raised in which generalized metric spaces that separability is equivalent to Lindelöf property. Obviously, separability is equivalent to Lindelöf property for stratifiable spaces. In Moore spaces, however, Lindelöf property implies separability, and there exists a separable Moore space which is not a Lindelöf space (for example, Niemytzki's tangent disc topology); moreover, whether every normal separable Moore space is Lindelöf is independent of the axioms of set theory ([3]). So the relationship between separability and

Lindelöf property in Moore spaces is clear.

$\aleph$-spaces are of other class of generalized metric spaces. The following two problems raised:

(1) whether there exists a separable $\aleph$-space which is not an $\aleph_0$-space ?

(2) how are the relationships between normal separable $\aleph$-spaces and $\aleph_0$-spaces ?

In this paper, we construct a completely regular, separable $\aleph$-space which is not an $\aleph_0$ -space. More precisely, we show that whether every normal, separable $\aleph$-space is an $\aleph_0$-space is independent of the axioms of set theory.

Let X be a topological space, a collection $\mathcal{F}$ of subsets of the X is a k-network if whenever K is a compact subset of an open set U, there exists a finite $\mathcal{F}' \subset \mathcal{F}$ such that $K \subset \cup\mathcal{F}' \subset U$. A regular space with σ-locally-finite k-network is called an $\aleph$-space. A regular space with countable k-network is called an $\aleph_0$-space. $\aleph_0$-spaces are hereditarily Lindelöf and hereditarily separable.

2. Main results

Throughout this section Martin's Axiom plus the negation of the Continuum Hypothesis is abbreviated as (MA+$\neg$CH). $\mathbb{N}$ denotes the positive integers, R the real numbers, and Q the rational numbers.

Example 1. There exists a completely regular, separable $\aleph$-space which is not an $\aleph_0$-space.

Construction. Let X be subset of R such that $Q \subset X \subset R$ and $|X| > \omega$. Let $Y = X \cup (\bigcup_{n \in N} Q \times \{1/n\})$, and define a base $\mathcal{B}$ for the desired topology on Y as follows:

(1) if $y \in Y-X$, let $\{y\} \in \mathcal{B}$, and

(2) if $x \in X$, then

$$\{\{x\} \cup (\bigcup_{n \geq m} ([a_{x,n}, x) \cap Q) \times \{1/m\} \mid m \in N,\ x > a_{x,n} \in R \} \subset \mathcal{B}.$$

Y is easily seen to be a completely regular, separable space. Y is not an $\aleph_0$-space since X is a uncountably discrete closed subset of Y.

To see that Y is also an $\aleph$-space, it is sufficient to show that Y is a σ-space, and every compact subset of Y is finite. Y is a σ-space because it is the union of countable many closed metrizable spaces. If K is an infinite compact set of Y, $X \cap K$ and $(Q \times \{1/n\}) \cap K_{(n \in N)}$ are finite because X and $(Q \times \{1/n\})_{(n\in N)}$ are discrete closed subsets of Y. Since K is infinite, there are infinite many n such that $(Q \times \{1/n\}) \cap K \neq \emptyset$. Hence there are a subsequence (n_i) of N and a subset $\{x_i \mid i \in N\}$ of Y such that $x_i \in (Q \times \{1/n_i\}) \cap K$ for each $i \in N$. Since K is a compact metrizable space (every compact subset is metrizable in σ -space), without loss of generality, we can assume $x_i \to x_0 \in K$. Since the point in Y-X is isolated, $x_0 \in X$. Denote by $x_i = (r_i, 1/n_i)$, since

$$V = \{x_0\} \cup (\bigcup_{n\in N}([x_0-1), x_0) \cap Q) \times \{1/n\})$$

is a neighborhood of the point x_0, there exists a $j \in N$ such that $x_i \in V$ whenever $i \geq j$, i.e., $x_0 - 1 \leq r_i < x_0$. We can assume each $r_i < x_0$. Put

$$q_n = \begin{cases} (r_i + x_0)/2 & n=n_i \ ; \\ x_0 - 1 & n \neq n_i \ ; \ n, i \in N \ . \end{cases}$$

Then $U = \{x_0\} \cup (\bigcup_{n\in N}(q_n, x_0) \cap Q) \times \{1/n\})$ is a neighbourhood of the point x_0, and each $x_i \notin U$. This contradicts the $x_i \to x_0$. Hence every compact subset of Y is finite. Therefore Y is an $\aleph'$ -space.

Example 2. (MA+$\neg$CH) There exists a normal, separable $\aleph$-space which is not an $\aleph_0$-space.

Construction. Let Y is the space which is definited in

Example 1 such that $|X|<2^{\omega}$. It is sufficient to show that Y is normal under (MA+¬CH). Suppose the H and K are disjoint closed subsets of Y. Since points in Y-X are isolated, we can assume that $H \cup K \subset X$. From[2], under (MA+¬CH), every subset of X is a F_σ-set relative to X with the induced topology of real line R, therefore $H = \bigcup_{n\in N} H_n$ and $K = \bigcup_{n\in N} K_n$ where for each n, $H_n \subset H_{n+1}$, $K_n \subset K_{n+1}$, and each H_n and K_n is closed in X with the induced topology of R. Now, since real line R is hereditarily normal, and for each n, H_n, K_n is a pair of separated sets of R, there exists disjoint open sets V_n and U_n in R containing H_n and K_n, respectively. Put

$$V = H \cup (\bigcup_{n\in N}(V_n \cap Q) \times \{1/n\}) \text{, and}$$

$$U = K \cup (\bigcup_{n\in N}(U_n \cap Q) \times \{1/n\}) \text{,}$$

then V and U are disjoint open sets in Y containing H and K, respectively. To see these, we only need to show that V is open. For each $x \in V$, we can assume $x \in H$. Since $H = \bigcup_{n\in N} H_n$ where for each n, $H_n \subset H_{n+1}$, there exists a $n \in N$ such that $x \in H_m$ whenever $m \geq n$. For each $m \geq n$, since $x \in V_m$ where V_m is an open set in real line R, there exists $a_{x,m} \in R$ such that $a_{x,m} < x$, and $[a_{x,m}, x) \subset V_m$. Then

$$x \in \{x\} \cup (\bigcup_{m\geq n}([a_{x,m}, x) \cap Q) \times \{1/m\})$$
$$\subset \{x\} \cup (\bigcup_{m\geq n}(V_m \cap Q) \times \{1/m\}) \subset V.$$

Hence, V is an open set in Y. Y is normal.

Theorem Whether every normal, separable $\aleph$-space is an $\aleph_0$-space is independent of the axioms of set theory.

Proof. From [1], under Continuum Hypothesis, Jones shows that every normal, separable space is $\aleph_1$-compact (i.e., every uncountable subset has a limit point). Since each $\aleph_1$-compact $\aleph$-space is an $\aleph_0$-space, therefore under Continuum Hypothesis, every normal, separable $\aleph$-space is an $\aleph_0$-space.

From Example 2, under (MA+$\neg$CH), there exists a normal, separable $\aleph$-space which is not an $\aleph_0$-space.

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